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The Dirichlet problem in the class of sh_m-functions on a Stein manifolds X.

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1. Introduction

The theory of strongly $m-$subharmonic (sh_m-)functions has an important role in potential theory. It expands and develops the well-known pluripotential theory, created at the end of the last century, which at present is the main subject for studying analytic functions of several complex variables and plurisubharmonic functions.

The pluripotential theory is based on plurisubharmonic (psh) functions and is related to the Monge-Ampère operator $(dd^c u)^n$. Here, as usual $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - \overline{\partial}}{4i}$. This theory is based on research in numerous fundamental works of E. Bedford , A. Taylor, J. Siciak , A. Sadullaev and others (see, for example, [2, 10, 15]).

sh_m-functions are related to the operators

$$(dd^c u)^m \wedge \beta^{n-m}, \ \ 1 \leq m \leq n, \tag{1}$$

where is $\beta = dd^c |z|^2-$the standard volume form in the complex space $\mathbb{C}^n$.

Since $dd^c u \wedge \beta^{n-1} = \Delta u \beta^n$, operator (1) for $m = 1$ gives the Laplace operator, and for $m = n$ the Monge-Ampère operator. The operator (1) is called the complex operator in Hessians, because it is easy to calculate

$$(dd^c u)^m \wedge \beta^{n-m} = m! \, (n-m)! H_m \, (u) \, \beta^n,$$

where $H_m \, (u) \, = \sum\limits_{1 \leq j_1 < ... < j_m \leq n} \lambda_{j_1} ... \lambda_{j_m} -$ is the Hessian of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ of the matrix $(u_{j,\bar{k}})$.

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The complex operator in Hessians (1) was successfully considered in the works [3, 4, 6, 7]. In their studies, a class of sh_m-functions was defined in the class of integrable functions $L^1_{loc}(D)$. Potential theory in the class of sh_m-functions was constructed in [9].

The purpose of this paper is to introduce and study sh_m-functions on complex manifolds $X \subset \mathbb{C}^N$, $\dim X = n$, $n \leq N$. There are different ways to define sh_m-functions on complex manifolds: using local coordinates, using retraction $\pi : \mathbb{C}^N \to X$, using Jensen measures. In this paper, to define sh_m-functions on a complex manifold X, we use spaces of Jensen measures. Details about the Jensen measures with respect to the class of functions on a compact set $K \subset \mathbb{C}^N$ can be found in [13] (see also [1,8]).

2.   Hessians

In Section 2 of this paper, we present definition and some properties of sh_m- functions in the space $\mathbb{C}^n$. In Section 3 we define Jensen measures on compact sets $K \subset \mathbb{C}^N$ with respect to sh_m-functions and formulate the Edward's duality theorem in the class of sh_m- functions.

In Section 4 we define sh_m-functions in domains $D \subset X$ of a complex manifold $X \subset \mathbb{C}^N$, $\dim X = n$, $n \leq N$ and given some of their potential properties. In section 5 we define sh_m-functions in local coordinates of a manifold X and study their connection with differential forms.

Section 6 is a supplement to the work that introduces the definition and studied some properties of $m-$polar sets and $\mathcal{P}_m-$measures.

3.   Hessians

Let $u \in C^2(D)$ be a twice differentiable function given in a domain $D \subset \mathbb{C}^n$. The second-order differential $dd^c u = \frac{i}{2} \sum_{j,k} u_{j,\bar{k}} dz_j \wedge d\bar{z}_k$ is a Hermitian quadratic form, where $u_{j,\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$. Therefore, via appropriate unitary transformation of coordinates, it is reduced to a diagonal form $dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge d\bar{z}_1 + ... + \lambda_n dz_n \wedge d\bar{z}_n]$, where $\lambda_1, ..., \lambda_n-$the eigenvalues of the Hermitian matrix $(u_{j,\bar{k}})$.

It's clear that

$$(dd^c u)^k \wedge \beta^{n-k} = k!\,(n-k)!H_k(u)\,\beta^n, \quad k = 1,..,n,$$

where $H_k(u) = \sum_{1 \leq j_1 < ... < j_k \leq n} \lambda_{j_1} \cdot ... \cdot \lambda_{j_k}-$is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Definition 3.1 *(see [12])*. *A function $u \in C^2(D)$ is called sh_m in domain $D \subset \mathbb{C}^n$, if :*

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0 \quad \forall k = 1, 2, ..., n - m + 1.$$

It is known that for all twice differentiable sh_m-functions $u, v_1, ..., v_{n-m}$ it holds

$$dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{2}$$

Moreover, if a twice differentiable function u satisfies (2) for all twice differentiable sh_m-functions $v_1, ..., v_{n-m}$, then u is necessarily a sh_m-function. Using this we can define sh_m-functions in the class L^1_{loc}.

Definition 3.2 *(see [12]).* *A function* $u \in L^1_{loc}(D)$ *is called* sh_m *in domain* $D \subset \mathbb{C}^n$, *if it is upper semicontinuous and for any twice differentiable* sh_m-*functions* $v_1, ..., v_{n-m}$ *the current* $dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ *defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \right] (\omega) =$$

$$= \int u \, dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{(0,0)}$$

is positive.

It is clear that $psh = sh_1 \subset sh_2 \subset ... \subset sh_n = sh$ and the following important property holds

Theorem 3.3 *(see [12]).* *If* $u \in sh_m(D)$, *then for any complex hyperplane* $\Pi \subset \mathbb{C}^n$ *restriction* $u|_\Pi$ *is a* sh_m-*function in* $D \bigcap \Pi$, *i.e.*

$$u|_\Pi \in sh_m(D \cap \Pi).$$

4. Jensen measure and sh_m-functions on a compact set $K \subset \mathbb{C}^N$.

In the previous section, we studied sh_m-functions in domains $D \subset \mathbb{C}^n$. In this section, we introduce the notion of sh_m-functions on a compact set $K \subset \mathbb{C}^N$. In [2] (see also [3]), the definition $sh_m(K)$ is based on the class of $sh_m^{0,c}-$functions that are continuous sh_m-functions in some neighborhood of K.

Definition 4.1 *(see[2]).* *Let* K *be a compact set in* $\mathbb{C}^N$, $1 \le m \le N$, *and let* μ *be a non-negative regular Borel measure defined on* K *with* $\mu(K) = 1$. *We say that* μ *is a Jensen measure with the barycenter* $z \in K$, *w.r.t.* $sh_m^{0,c}(K)$ *if*

$$u(z) \le \int_K u \, d\mu, \quad \forall u \in sh_m^{0,c}(K).$$

The set of such measures will be denoted by $\Im^m_z(K)$.

Note that if the measure μ, defined on K, $\mu(K) = 1$ is a Jensen measure with the barycenter at the point $z \in K$, then

$$u(z) \le \int_K u \, d\mu, \quad \forall u \in sh_m^0(K),$$

where $sh_m^0(K) -$ the class of functions that are sh_m-functions (not necessarily continuous, i.e. $sh_m^0(K) \supset sh_m^{0,c}(K)$) in some neighborhood of K.

Using the Jensen measures, we now define the notion of sh_m-functions on a compact set K.

Definition 4.2 *(see [2]).* *Let* K *be a compact set in* $\mathbb{C}^N$. *An upper-semicontinuous function* u *defined on* K *is said to be* sh_m-*function on* K, $1 \le m \le N$, *if*

$$u(z) \le \int_K u \, d\mu, \quad \forall z \in K, \ \ \forall \mu \in \Im^m_z(K).$$

The set of all sh_m-functions on K will be denoted by $sh_m(K)$. sh_m-functions on compact sets share a lot of basic properties with sh_m-functions on open sets. Some of these properties are listed below (see [2]).

Theorem 4.3 *Let K be a compact set in $\mathbb{C}^N$ and $1 \leq m \leq N$. Then*

1) if $u, v \in sh_m(K)$, then $su + kv \in sh_m(K)$, for all $s, k \geq 0$;

2) if $u, v \in sh_m(K)$, then $\max\{u, v\} \in sh_m(K)$;

3) if $u_j \in sh_m(K)$ is a decreasing sequence, then $u = \lim\limits_{j \to \infty} u_j \in sh_m(K)$, provided $u(z) >$
$-\infty$ for some point $z \in K$;

4) If $u \in sh_m(K)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a convex and non-decreasing function, then $\gamma \circ u \in$
$sh_m(K)$.

Let us present also the Edwards theorem (see [14]), which is often used in the study of the class $sh_m(K)$.

Theorem 4.4 *(see [2]).* *Let K be a compact subset in $\mathbb{C}^N$ and $1 \leq m \leq N$, and let $g(z)$ be a lower semicontinuous function defined on K. Then we have*

$$\sup\left\{ u(z): \quad u(z) \in sh_m^{0,c}(K), \quad u \leq g \right\} = \inf\left\{ \int\limits_K g d\mu : \quad \mu \in \Im_z^m \right\}.$$

Using this theorem, we prove the following important theorem.

Theorem 4.5 *(see [2]).* *Let K be a compact set in $\mathbb{C}^N$ and $1 \leq m \leq N$.*

a) Let u be an upper semicontinuous function on K. Then it sh_m-function $u \in sh_m(K)$ if and only if there exists a monotonically increasing sequence $u_j \in sh_m^{0,c}(K)$ such that $u_j \nearrow u$ on K.

b) $u \in sh_m(K)$ if and only if there is a monotonically decreasing sequence $u_j \in sh_m^{0,c}(K)$ such that $u_j \searrow u$.

5. sh_m-functions on a manifold X.

Suppose $X \subset \mathbb{C}^N$ is a complex manifold of complex dimension n and let $O(X)$ denote the ring of holomorphic functions on X.

Definition 5.1 *(see [17]).* *A complex analytic manifold X of dimension n which is countable at infinity is said to be a Stein manifold if*

1) X is holomorph-convex, that is,

$$\hat{K} = \left\{ z: \quad z \in X, \ |f(z)| \leq \sup_K |f| \ \text{ for all } f \in \mathcal{O}(X) \right\}$$

is a compact subset of X for every compact subset $K \subset X$;

2) If z_1 and z_2 are different points in X, then $f(z_1) \neq f(z_2)$ for some $f \in \mathcal{O}(X)$;

3) For every $z \in X$, one can find n functions $f_1, ..., f_n \in \mathcal{O}(X)$ which form a coordinate system at z.

Let $K \subset X$ some compact subset. Then there has a class $sh_m^{0,c}(K)$ which using to define classes of Jensen measures. By the section 3 we know how to define sh_m-functions on $\mathrm{K} \subset \mathbb{C}^N$ compact sets. As analogue of definition 3.2 we can define sh_m-functions on $K \subset X$ compact sets.

Let $K \subset X$ some compact subset $(\dim_c K = \dim_c X = n)$ and $sh_m^{0,c}(K)$ the class of functions that are continuous sh_m-functions in some neighborhood of K.

Definition 5.2 *Let μ be a non-negative regular Borel measure defined on K with $\mu(K) = 1$. We say that μ is a Jensen measure with the barycenter $z \in K$, w.r.t. $sh_m^{0,c}(K)$ if*

$$u(z) \leq \int\limits_K u \, d\mu, \qquad \forall u \in sh_m^{0,c}(K).$$

Definition 5.3 *An upper-semicontinuous function u defined on K is said to be sh_m-function on K, $1 \leq m \leq n$, if*

$$u(z) \leq \int\limits_K u \, d\mu, \qquad \forall z \in K, \quad \forall \mu \in \Im_z^m(K).$$

Theorem 3.3 and **Theorem 3.5** are holds in this class too. Properties 1) and 2) from **Theorem 3.3** are obviously. In proofs of others follows easily from the properties of Borel measures, Choquet lemma and from Edwards theorem.

Definition 5.4 *A function $u(z)$ given in D is called sh_m in D, $(u \in sh_m(D))$ if:*
1) it is upper semicontinuous in D, i.e. $u(z) \geq \overline{\lim\limits_{w \to z}} u(w)$.
2) for all compact subset $K \subset D$ the restriction $u|_K$ is sh_m in K, i.e. $u|_K \in sh_m(K)$.

The class of sh_m-functions in D denoted by $sh_m(D)$. As usual, the function $u(z) \equiv -\infty$ is also included in this class. The following properties of the class $sh_m(D)$ can be shown easily by the properties of sh_m-functions on compact sets.
1) *Linear combination of sh_m-functions with non-negative coefficients is a sh_m-function, i.e.*

$$u_k(z) \in sh_m(D), \quad a_k \in \mathbb{R}^+ \ (k = 1, 2, ..., p) \quad \Rightarrow \quad a_1 u_1(z) + a_2 u_2(z) + ... + a_p u_p(z) \in sh_m(D);$$

2) *Maximum of a finite number of sh_m-functions is also a sh_m-function, i.e.*

$$u_1(z), u_2(z), ..., u_p(z) \in sh_m(D) \quad \Rightarrow \quad \max\{u_1(z), u_2(z), ..., u_p(z)\} \in sh_m(D);$$

3) The following relation is true

$$sh_1(D) \subset ... \subset sh_m(D) \subset ... \subset sh_n(D);$$

4) Limit of a monotonically decreasing sequence of sh_m-functions is also a sh_m-function, i.e.

$$u_j(z) \in sh_m(D), \quad u_j(z) \geq u_{j+1}(z) \ (j = 1, 2, ...) \quad \Rightarrow \quad \lim\limits_{j \to \infty} u_j(z) \in sh_m(D);$$

5) A uniformly convergent sequence of sh_m-functions converges to a sh_m-function, i.e.

$$u_j(z) \in sh_m(D), \ (j = 1, 2, ...), \ u_j(z) \rightrightarrows u(z) \quad \Rightarrow \quad u(z) \in sh_m(D);$$

6) If $u_j(z) \in sh_m(D), \ u_j(z) \le M_j, \ (j = 1, 2, ...)$ and series $\sum_{j=1}^{\infty} M_j$ converges, then $\lim_{n \to \infty} \sum_{j=1}^{n} u_j(z)$ converges to a sh_m-function.

7) Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a convex and non-decreasing function and $u(z) \in sh_m(D)$. Then $\gamma \circ u \in sh_m(D)$.

6. sh_m-functions on local coordinates in X.

Below it is useful for us to define sh_m-functions using the volume form on X. On this purpose, we take the restriction of the volume form $dd^c \|z\|^2 = \frac{i}{2} [dz_1 \wedge d\bar{z}_1 + ... + dz_N \wedge d\bar{z}_N]$ on X and let's put $\beta = dd^c \|z\|^2 |_X$. In local coordinates $\varphi(\xi): \ B \to U, \ B \subset \mathbb{C}^n, \ U \subset X$, differential form of bidegree $(1, 1)$ has form

$$\beta = \frac{i}{2} [d\varphi_1(\xi) \wedge d\bar{\varphi}_1(\xi) + ... + d\varphi_N(\xi) \wedge d\bar{\varphi}_N(\xi)].$$

Differential form β is positive and closed. Let u is a twice differentiable function given in X. Then its differential form $dd^c u$ be a differential form of bidegree $(1, 1)$. In local coordinates $\varphi(\xi): \ B \to U$ is calculated as

$$dd^c u = dd^c u(\varphi(\xi)) = \sum_{j,t=1}^{n} \frac{d^2 u(\varphi(\xi))}{\partial \xi_j \partial \bar{\xi}_t} \partial \xi_j \wedge \partial \bar{\xi}, \ \xi \in \mathbb{C}^n.$$

After these reminders, we can determine sh_m-functions on X using the definition in complex space $\mathbb{C}^n$.

Definition 6.1 *Function $u \in C^2(D)$ called sh_m-function in the area $D \subset X$, in D have places*

$$(dd^c u)^k \wedge \beta^{n-k} \ge 0, \ k = 1, 2, ..., n - m + 1. \tag{3}$$

Note that in local coordinates $\varphi(\xi): \ B \to U, \ B \subset \mathbb{C}^n, \ U \subset X$, differential forms (3) leads to following

$$(dd^c u)^k \wedge \beta^{n-k} = \left\{ \sum_{s,t=1}^{n} \frac{d^2 u(\varphi(\xi))}{\partial \xi_s \partial \bar{\xi}_t} \partial \xi_s \wedge \partial \bar{\xi} \right\}^k \wedge$$

$$\wedge \left\{ \frac{i}{2} [d\varphi_1(\xi) \wedge d\bar{\varphi}_1(\xi) + ... + d\varphi_N(\xi) \wedge d\bar{\varphi}_N(\xi)] \right\}^{n-k} \ge 0, \ k = 1, 2, ..., n - m + 1.$$

There is the following property relating sh_m-functions with sh_m-functions in complex space $\mathbb{C}^N$.

Theorem 6.2 *If $w(z)$ is a sh_m-function in some neighborhood $U \subset \mathbb{C}^N, \ w(z) \in C^2(U)$, then the restriction $u = w(z)|_X$ is a sh_m-function in $D = U \bigcap X$, i.e. $u = w|_X \in sh_m(U \bigcap X)$.*

Proof. Fixing some point $o \in D \subset U$ and get tangent space $T_o X$. In some neighborhood $B = B(o, r)$ projection $\pi : X \bigcap B \to T_o X$ be a biholomorphic mapping and besides that $\pi^{-1} : B \bigcap T_0 X \to B \bigcap X$ be a local coordinate on X.

We have $w \in sh_m(U) \bigcap C^2(U)$, then by Theorem 2.3 the restriction $w|_{T_o X} \in sh_m(B \bigcap T_o X)$ is sh_m−function, i.e.

$$\left(dd^c w|_{T_o X}\right)^k \wedge \left[dd^c \|\xi\|^2|_{T_o X}\right]^{n-k} \geq 0, \ \ k = 1, 2, ..., n - m + 1.$$

But then

$$\left(dd^c w \circ \pi^{-1}\right)^k \wedge \left[dd^c \left\|\pi^{-1}\right\|^2\right]^{n-k} \geq 0, \ \ k = 1, 2, ..., n - m + 1.$$

On the other side, since $\pi : X \bigcap B \to T_o X$ is a projection, then at a point $o \in X$ holds $dd^c \left\|\pi^{-1}\right\|^2|_o = \beta|_o$. It means that $w|_X \in sh_m(B \bigcap X)$ and therefore $w|_X \in sh_m(D)$. The proof is complete.

For functions on a manifold is true the inequality (2).

Theorem 6.3 *Let u is a twice differentiable sh_m−function in D. Then for all twice differentiable sh_m−functions $v_1, ..., v_{n-m} \in sh_m(D) \bigcap C^2(D)$ it holds:*

$$dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{4}$$

Conversely is holds too, i.e. if a twice smooth function u, satisfies this relation for all $v_1, ..., v_{n-m} \in sh_m(D) \bigcap C^2(D)$, then u certainly is sh_m−function. Moreover, it is enough to see restriction on X class of second-order polynomials of the form

$$v_j = \sum_{k=1}^{n} c_{j,k} |z_k|^2 \in sh_m(\mathbb{C}^n), \ c_{j,k} - const.$$

Proof. By definition of sh_m−functions, it follows that in local coordinates $\varphi(\xi) : B \to X, \ B \subset \mathbb{C}^n$, differential forms become to the next form

$$(dd^c u)^k \wedge \beta^{n-k} = \left\{\sum_{s,t=1}^{n} \frac{d^2 u(\varphi(\xi))}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right\}^k \wedge \tag{5}$$

$$\wedge \left\{\frac{i}{2} \left[d\varphi_1(\xi) \wedge d\bar{\varphi}_1(\xi) + ... + d\varphi_N(\xi) \wedge d\bar{\varphi}_N(\xi)\right]\right\}^{n-k} \geq 0, \ \ k = 1, 2, ..., n - m + 1.$$

The same relation is true for each of the functions $v_1, ..., v_{n-m} \in sh_m(D) \bigcap C^2(D)$:

$$(dd^c v_j)^k \wedge \beta^{n-k} = \left\{\sum_{s,t=1}^{n} \frac{d^2 v_j(\varphi(\xi))}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right\}^k \wedge$$

$$\wedge \left\{\frac{i}{2} \left[d\varphi_1(\xi) \wedge d\bar{\varphi}_1(\xi) + ... + d\varphi_N(\xi) \wedge d\bar{\varphi}_N(\xi)\right]\right\}^{n-k} \geq 0, \ \ k = 1, 2, ..., n-m+1, \ j = 1, 2, ..., n-m.$$

So the functions $u, v_1, ..., v_{n-m} \in sh_m(D) \bigcap C^2(D)$ are sh_m−functions with respect to strictly positive d−closed form $\alpha = \frac{i}{2} \left[d\varphi_1(\xi) \wedge d\bar{\varphi}_1(\xi) + ... + d\varphi_N(\xi) \wedge d\bar{\varphi}_N(\xi)\right]$. According to the notes above after the formulation of Theorem 5.3 it is true

$$\left[\sum_{s,t=1}^{n} \frac{d^2 u\left(\varphi\left(\xi\right)\right)}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right] \wedge \left[\sum_{s,t=1}^{n} \frac{d^2 v_1\left(\varphi\left(\xi\right)\right)}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right] \wedge \ldots$$

$$\wedge \left[\sum_{s,t=1}^{n} \frac{d^2 v_{n-m}\left(\varphi\left(\xi\right)\right)}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right] \wedge \alpha^{m-1} \geq 0,$$

which is equivalent to $dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0$. Doing these calculations from the bottom up, we get that from

$$dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0 \quad \forall v_1, \ldots, v_{n-m} \in sh_m\left(D\right)\bigcap C^2\left(D\right)$$

it follows that the function $u\left(\varphi\left(\xi\right)\right)$ is sh_m-function in local coordinates with respect to the strictly positive $d-$closed form $\alpha = \frac{i}{2}\left[d\varphi_1\left(\xi\right) \wedge d\bar{\varphi}_1\left(\xi\right) + \ldots + d\varphi_N\left(\xi\right) \wedge d\bar{\varphi}_N\left(\xi\right)\right]$:

$$\left(dd^c u\right)^k \wedge \beta^{n-k} = \left\{\sum_{s,t=1}^{n} \frac{d^2 u\left(\varphi\left(\xi\right)\right)}{\partial \xi_s \partial \bar{\xi}_t} d\xi_s \wedge d\bar{\xi}_t\right\}^{k} \wedge$$

$$\wedge \left\{\frac{i}{2}\left[d\varphi_1\left(\xi\right) \wedge d\bar{\varphi}_1\left(\xi\right) + \ldots + d\varphi_N\left(\xi\right) \wedge d\bar{\varphi}_N\left(\xi\right)\right]\right\}^{n-k} \geq 0, \quad k = 1, 2, \ldots, n-m+1.$$

It means that $u \in sh_m\left(D\right)$. The proof is complete.

Definition 6.4 *Function $u \in L^1_{loc}\left(D\right)$ is called sh_m-function in domain $D \subset X$, if it is upper semicontinuous and for any twice differentiable sh_m-functions $v_1, \ldots, v_{n-m} \in sh_m\left(D\right)\bigcap C^2\left(D\right)$ current $dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ is defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right]\left(\omega\right) = \int u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0}$$

is positive.

Here $\beta = dd^c \|z\|^2 |_X$ and $F^{0,0}$ is a class of test functions (see [16]).

7. $m-$polar sets and $\mathcal{P}_m-$measure on X.

The concepts of $m-$polar sets and P_m- measures on X are the basic concepts of potential theory in the class of sh_m-functions on X as they are for sh_m-functions on the complex space $\mathbb{C}^n$ (see, for example, [16]).

$m-$polar sets on X are also defined as $m-$polar sets in the complex space $\mathbb{C}^n$.

Definition 7.1 *A set $E, E \subset D \subset X$ is called $m-$polar in D, if there exist a function $u(z) \in sh_m(D)$, $u(z) \not\equiv -\infty$, such that $u|_E = -\infty$.*

By the definition 5.2 it follows that the Hausdorff measure of $m-$polar set $H_{2n-2+\varepsilon}(E) = 0 \ \forall \varepsilon > 0$. In addition, the countable union of $m-$polar sets is also $m-$polar set, i.e. if $E_j \subset D$ are $m-$polar sets, then the set $E = \bigcup_{j=1}^{\infty} E_j$ is also $m-$polar. It is clear that if the set $E \subset D$ is $m-$polar, then it is also $k-$polar set such that $k = m+1, \ m+2, \ldots, n$.

Definition 7.2 *A domain $D \subset X$ is called $m-$regular if there exists a function $\rho\left(z\right) \in sh_m\left(D\right)$, $\rho\left(z\right) < 0$ such that $\lim_{z \to \partial D} \rho\left(z\right) = 0$.*

The following theorem is useful in the study of $\mathcal{P}_m-$measures, which is similarly proved in the case of sh_m measures on $\mathbb{C}^N$.

Theorem 7.3 *Let $D \subset X$ is a m-regular domain and subset $E \subset D$ is such that set $E \cap G$ is $m-$polar in G for any compact subdomain $G \subset\subset D$. Then the set E is also $m-$polar in D. In addition, there is a function $u(z) \in m - sh(D)$, $u\,|_D < 0$, $u- \not\equiv \infty$ such that $u\,|_E \equiv -\infty$.*

Let $E \subset D$ is a subset of $m-$regular domain $D \subset X$ and $1 \le m \le n$.

Definition 7.4 *Consider the class of functions*

$$\mathcal{U}(E, D) = \{u(z) \in sh_m(D): \ u\,|_D \le 0, \ u\,|_E \le -1\}$$

and put $\omega(z, E, D) = \sup\{u(z): \ u \in U(E, D)\}$. Then the regularization $\omega * (z, E, D)$ is called $\mathcal{P}_m-$measure of the set E with respect to the domain D.

It is clear, that $\omega * (z, E, D) \in m - sh(D)$. Because, by the Choquet lemma, there exists a countable subfamily $U' \subset U(E, D)$ such that $\{\sup\{u(z)\}: \ u(z) \in U'(E, D)\}^* \equiv \omega * (z, E, D)$. Thus, it follows that P_m-measure $\omega*(z, E, D)$ can be represented as the limit of a monotonically increasing sequence $\{u_j(z)\} \subset U(E, D) : [\lim_{j \to \infty} u_j(z)]^* \equiv \omega * (z, E, D)$.

$\mathcal{P}_m-$measure has the following simple properties:

1) *(monotonicity) if $E_1 \subset E_2$, then $\omega * (z, E_1, D) \ge \omega * (z, E_2, D)$;*

If $E \subset D_1 \subset D_2$, then $\omega * (z, E, D_1) \ge \omega * (z, E, D_2)$;

2)$\mathcal{P}_m-$*measure $\omega * (z, E, D)$ is either nowhere equal to zero, or identically equal to zero. $\omega * (z, E, D) \equiv 0$ if and only if E is $m-$polar set in D (proved similarly using Theorem 6.3.);*

3) *(theorem about two constants). If a function $u(z)$ is $m-$subharmonic function in $D \subset \mathbb{C}^n$ and $u\,|_D \le M$, $u\,|_E \le m$, $(E \subset D)$, then following inequality*

$$u(z) \le M \cdot (1 + \omega * (z, E, D)) - m \cdot \omega * (z, E, D)$$

holds for all $z \in D$ (obviously).

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Задача Дирихле в классе sh_m-функций на многообразии X.

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Данная статья посвящена определению и изучению сильно $m-$субгармонических функций на комплексных многообразиях. Вводится определение сильно $m-$субгармонических функций на многообразии Штейна и доказываются некоторые основные свойства.

Ключевые слова: $m-$субгармонические функции на комплексном пространстве, Мера Иенсена, Многообразие Штейна, полярное множество.