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C^k-Properties of Subharmonic Functions

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Abstract—This paper presents the results on differential properties of the subharmonic functions. It is proved that the Riesz potential $U^{\mu}_{n-\alpha}(x)$ $(1 \le \alpha < n)$ belongs to the Zygmund class under certain sufficient conditions imposed on the measure μ.

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1. INTRODUCTION

The well-known theorem of Cartan [1] (see also [2, p. 231] on the continuity of Riesz potentials

$$U^{\mu}_{n-\alpha}(x) = \int \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n,$$

states that for any $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \mathbb{R}^n$ of capacity $C_{n-\alpha}(G_\varepsilon) < \varepsilon$, such that $U^{\mu}_{n-\alpha}(x)$ is continuous in the complement $\mathbb{R}^n$. Note that Cartan's theorem is also true in the case of a logarithmic potential

$$U^{\mu}_{0}(x) = \int \ln|x-y| d\mu(y), \quad n = 2.$$

This theorem is an analogue of Luzin's theorem on the almost everywhere continuity (C-property) of measurable functions. According to the theorem on the Riesz representation for subharmonic functions (see, for example, [2, 3]) and Cartan's theorem, for a subharmonic function $u(x)$ in the domain $D \subset \mathbb{R}^n$ there exists an open set $G_\varepsilon \subset D$, with Newtonian capacity $C_{n-2}(G_\varepsilon) < \varepsilon$ such that $u(x)$ is continuous in the complement $D \setminus G_\varepsilon$ for any $\varepsilon > 0$.

In the work of Sadullaev and Madrakhimov [4], the smoothness of subharmonic functions outside a set thicker than a set of small Newtonian (logarithmic) capacity is established. It was shown that for any $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D$ such that the Hausdorff coverage of $\hat{H}_{n-2+p}(G_\varepsilon) < \varepsilon$, $0 < p \le 2$ and $u(x)$ belongs to the Lipschitz class on compact subsets of the difference $D \setminus G_\varepsilon$.

In [5, 6], the smoothness of subharmonic functions was studied on the basis of the theory of Calderon–Zygmund singular integrals [7, 8] and potential theory, which made it possible to obtain relatively definitive results. For a subharmonic function $u(x)$ defined in the domain $D \subset \mathbb{R}^n$, it was shown that for any $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D$ with Lebesgue measure $m(G_\varepsilon) < \varepsilon$, such that $u(x)$ belongs to the class C^2 of twice continuously differentiable functions on compact subsets of the difference $D \setminus G_\varepsilon$. For an analytic continuation of functions of several variables that are $\mathbb{R}$-analytic

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along a fixed direction (see [9]). Obtaining estimates for subharmonic functions plays an important role in the theory of potentials. Depending on the potential and the considered sets, various estimates were obtained for subharmonic functions (see, for example, [10–12]).

A subharmonic function, in general, does not belong to the Lipschitz class Lip_1 outside the set of small Hausdorff coverage $\hat{H}_{n-1}$. The question arises: what is the description of the "thinnest" set outside which a subharmonic function belongs to the class Lip_1 and is defined as the set of small Hausdorff coverage $\hat{H}_{n-1+0}$, where $\hat{H}_{n-1+0} = \sup_{\varepsilon>0} \hat{H}_{n-1+\varepsilon}$. However, there is a stronger statement than this, for any $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D$ with capacity $C_{n-1}(G_\varepsilon) < \varepsilon$, such that $u(x)$ defined on the domain $D \subset \mathbb{R}^n$ belongs to the C^1-class of continuously differentiable functions on compact subsets of the difference $D \setminus G_\varepsilon$.

It is known that a set of small C_{n-1}-capacity is "thinner" than a set of small Hausdorff coverage $\hat{H}_{n-1+0}$, but "thicker" than a set of small Hausdorff coverage $\hat{H}_{n-1}$ (see [2, 3, 13]).

An important subclass of the set of subharmonic functions is the class of convex functions. Studying properties of convex functions and their differentiability has been and still is one of the main subjects in the theory of convex surfaces and convex analysis [14–21]. Reshetnyak in [14] showed almost everywhere twice differentiability of convex functions, i.e., if $u(x)$ is a convex function in the domain $D \subset \mathbb{R}^n$, then

$$u(x+h) - u(x) - \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} h_i - \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} h_i h_k = 0\left(|h|^2\right),$$

almost at all points $x \in D$. In general, this statement does not hold in the case of subharmonic functions. For example, for a subharmonic function that becomes $-\infty$ on an everywhere dense set of points, such a statement is not true. In a recent paper [21], it was shown that for a convex domain $D \subset \mathbb{R}^n$, locally strictly convex function $u : D \to \mathbb{R}$ and for any continuous function $\varepsilon(x) : D \to (0, 1]$, there exists a locally strictly convex function $\vartheta(x)$ from the class $C^2(D)$, such that $m\{x \in D : u(x) \neq \vartheta(x)\} < \varepsilon_0$ for any $\varepsilon_0 > 0$ and $|u(x) - \vartheta(x)| < \varepsilon(x)$, $\forall x \in D$. Furthermore, it was proven that for the Hausdorff measure, it holds that $H_n(\Gamma_u \vartriangle \Gamma_\vartheta) < \varepsilon_0$, $\Gamma_u = (x, u(x)) \in \mathbb{R}^{n+1} : x \in D$. Some more results on Riesz potentials and on their differentiability with respect to capacities were given in the works [22–24].

2. CAPACITY, COVERAGE, AND HAUSDORFF MEASURE

2.1. Capacity

First, we define $C_{n-\alpha}-$capacity $(0 < \alpha < n)$ (see [2]). By $\mathfrak{M}_n$, we denote the set of all Borel measures in $\mathbb{R}^n, n \geq 2$, and let

$$U^\mu_{n-\alpha}(x) = \int \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n.$$

The capacity of a compact set $K \subset \mathbb{R}^n$ is denoted by $C_{n-\alpha}$, i.e.,

$$C_{n-\alpha}(K) = \sup\left\{\mu(\mathbb{R}^n) : \mu \in \mathfrak{M}_n,\, \mathrm{supp}\mu \subset K; U^\mu_{n-\alpha}(x) \leq 1,\, x \in \mathrm{supp}\mu\right\}.$$

For an arbitrary set $E \subset \mathbb{R}^n$, the quantities

$$\underline{C}_{n-\alpha}(E) = \sup\{C_{n-\alpha}(K) : \quad K \text{ is compact}, \quad K \subset E\}$$

and

$$\overline{C}_{n-\alpha}(E) = \inf\{C_{n-\alpha}(G) : \quad G \text{ is open}, \quad E \subset G\}$$

are called, respectively, internal and external capacities. If $\underline{C}_{n-\alpha}(E) = \overline{C}_{n-\alpha}(E)$, then the set E will be called C-measurable (measurable by capacity) and we will write $C_{n-\alpha}(E)$ instead of $\underline{C}_{n-\alpha}(E)$ or $\overline{C}_{n-\alpha}(E)$.

It is known that all Borel sets are $C-$measurable (see [2, p. 196]. Let us indicate some properties:

1) For any finite or countable family of $C-$measurable sets E_i, such that $E \subset \bigcup_i E_i$, we have

$$C_{n-\alpha}(E) \leq \sum_i C_{n-\alpha}(E_i).$$

2) $C_{n-\alpha}(B(x,r)) = A(n,\alpha) r^{n-\alpha}$, where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, where $A(n,\alpha)$ is a constant depending only on n and α.

2.2. Coverage and Hausdorff Measure

Let E be a set from $\mathbb{R}^n$. Consider a covering of set E by a finite or countable set of convex sets V_i with diameter V_i and set

$$\hat{H}_\alpha(E) = \inf \sum_i d_i^\alpha,$$

where inf on the right-hand side is taken over all such coverings. $\hat{H}_\alpha(E)$ is called a Hausdorff coverage of order α. We denote the corresponding exact lower bound by $H_\alpha^\varepsilon(E)$. Then H_α^ε is an everywhere finite and non-increasing function of the variable ε, i.e., $H_\alpha^{\varepsilon_1}(E) \geq H_\alpha^{\varepsilon_2}(E)$ for $\varepsilon_1 < \varepsilon_2$. The limit $H_\alpha(E) = \lim_{\varepsilon\to 0} H_\alpha^\varepsilon(E)$ is called the outer α-Hausdorff measure of the set E. Note that in this definition, instead of convex sets, it is sufficient to consider balls.

Below, we present some properties of the quantities $\hat{H}_\alpha$ and H_α that we will use

1) $\hat{H}_\alpha(E) \leq H_\alpha(E)$;

2) $\hat{H}_\alpha(E)$ and $H_\alpha(E)$ vanish simultaneously;

3) $\hat{H}_\alpha(E) \leq \sum_i \hat{H}_\alpha(E_i)$ and also $H_\alpha(E) \leq \sum_i H_\alpha(E_i)$ for any finite or countable family $\{E_i\}$ such that $E \subset \bigcup_i E_i$. The set $E \subset \mathbb{R}^n$ is called H_α-measurable if

$$H_\alpha(E) = H_\alpha\left(E \bigcap A\right) + H_\alpha(A \setminus E)$$

for any $A \subset \mathbb{R}^n$. If the set E is H_α-measurable, then $H_\alpha(E)$ is called the Hausdorff measure of E.

4) Let $E_1, E_2, ..., E_i, ...$ be pairwise disjoint H_α-measurable sets, then

$$H_\alpha\left(\bigcup_i E_i\right) = \sum_i H_\alpha(E_i),$$

i.e., H_α is a countably additive measure;

5) Every Borel set is H_α-measurable.

Now, we present two lemmas.

Lemma 1. *For any C-measurable set $E \subset \mathbb{R}^n$ and for any $\varepsilon > 0$, the inequality holds*

$$a_1\left[\hat{H}_{n-\alpha+\varepsilon}(E)\right]^{\frac{n-\alpha}{n-\alpha+\varepsilon}} \leq C_{n-\alpha}(E) \leq a_2 \hat{H}_{n-\alpha}(E),$$

where $a_1 > 0$ and $a_2 > 0$ are constants depending only on n, α and ε. Moreover, there exist sets E_1 and E_2 such that for any $\varepsilon > 0$ $\hat{H}_{n-\alpha+\varepsilon}(E_1) = 0$, $C_{n-\alpha}(E_1) > 0$ and $C_{n-\alpha}(E_2) = 0$, $\hat{H}_{n-\alpha}(E_2) > 0$.

In fact, this is a well-known lemma. However, it has never been formulated in this form. Inequality

$$a_1\left[\hat{H}_{n-\alpha+\varepsilon}(E)\right]^{\frac{n-\alpha}{n-\alpha+\varepsilon}} \leq C_{n-\alpha}(E)$$

follows from the following statement: let $E \subset \mathbb{R}^n$ be a set for which $0 < \overline{C}_\alpha(E) < \infty$. Then, for any $\varepsilon > 0$ it can be covered by a system of balls of radius r_i, where

$$a(n,\alpha,\varepsilon)\left(\sum_i r_i^{n-\alpha+\varepsilon}\right)^{\frac{n-\alpha}{n-\alpha+\varepsilon}} \leq \overline{C}_{n-\alpha}(E)$$

(see [2, p. 253]), and inequality

$$C_{n-\alpha}(E) \leq \hat{H}_{n-\alpha}(E)$$

follows from the above-mentioned properties 1) and 2) of $C_{n-\alpha}$-capacity. For the existence of a set E_1 and E_2, satisfying the last statement (see [13]).

Lemma 2. *Let μ be a finite Borel measure with compact support supp$\mu \subset \mathbb{R}^n$. Then, for any $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \mathbb{R}^n$ with capacity $C_{n-\alpha}(G_\varepsilon) < \varepsilon$, such that*

I) *the family of functions*

$$\int\limits_{|x-y|<\delta} \frac{d\mu(y)}{|x-y|^{n-\alpha}}$$

converges uniformly to zero as $\delta \to 0$ *on the set* $\mathbb{R}^n \setminus G_\varepsilon$;

II) *the family of functions*

$$\frac{\mu(B(x,t))}{t^{n-\alpha}}$$

converges uniformly to zero as $t \to 0$ *on the set* $\mathbb{R}^n \setminus G_\varepsilon$.

Proof. I) Since

$$\int\limits_{|x-y|<\delta} \frac{d\mu(y)}{|x-y|^{n-\alpha}} = \int \frac{d\mu(y)}{|x-y|^{n-\alpha}} - \int\limits_{|x-y|>\delta} \frac{d\mu(y)}{|x-y|^{n-\alpha}},$$

then it suffices to show that outside some set with small C_α-capacity, the family of functions

$$\int\limits_{|x-y|\geq\delta} \frac{d\mu(y)}{|x-y|^{n-\alpha}} \tag{1}$$

uniformly converges to the potential $U^\mu_{n-\alpha}(x)$ as $\delta \to 0$.

It is obvious that for each $\delta > 0$, the function (1) is continuous in x in $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$, it is monotonically increasing and converges to the potential $U^\mu_{n-\alpha}(x)$ as $\delta \to 0$. In addition, for any $\varepsilon > 0$, as Cartan's theorem ([2, p. 231]) asserts, there exists an open set $G_\varepsilon \subset \mathbb{R}^n$ with capacity $C_{n-\alpha}(G_\varepsilon) < \varepsilon$ such that the potential $U^\mu_{n-\alpha}(x)$ is continuous on the set $\mathbb{R}^n \setminus G_\varepsilon$. Therefore, according to Dini's theorem (on uniform convergence, see, for example, [25]) and the compactness of the support of the measure μ, the family of functions (1) uniformly converges to the potential $U^\mu_{n-\alpha}(x)$ as $\delta \to 0$ on the set $\mathbb{R}^n \setminus G_\varepsilon$.

II) Consider the equality

$$\frac{\mu(B(x,\delta_0))}{\delta_0^{n-\alpha}} + \int\limits_{\delta_0}^{\delta} \frac{d\mu(B(x,t))}{t^{n-\alpha}} = \frac{\mu(B(x,\delta))}{\delta^{n-\alpha}} + (n-\alpha)\int\limits_{\delta_0}^{\delta} \frac{\mu(B(x,t))}{t^{n-\alpha+1}}dt,$$

where $\delta > \delta_0 > 0$. This equality is obtained as a result of integration by parts of the integral located on the left-hand side. Considering the lower limit in this equality as $\delta_0 \to 0$, we obtain

$$\varliminf_{\delta_0 \to 0} \frac{\mu(B(x,\delta_0))}{\delta_0^{n-\alpha}} + \int\limits_{0}^{\delta} \frac{d\mu(B(x,t))}{t^{n-\alpha}} = \frac{\mu(B(x,\delta))}{\delta^{n-\alpha}} + (n-\alpha)\int\limits_{0}^{\delta} \frac{\mu(B(x,t))}{t^{n-\alpha+1}}dt. \tag{2}$$

Since the integral on the left side of equality (2) is the remainder of an improper integral (potential)

$$U^\mu_{n-\alpha}(x) = \int\limits_0^\infty \frac{d\mu(B(x,t))}{t^{n-\alpha}},$$

then outside the union of the sets

$$S_1 = \left\{x : \varliminf_{t \to 0} \frac{\mu(B(x,t))}{t^{n-\alpha}} = +\infty\right\}$$

and

$$S_1 = \{x : U^\mu_{n-\alpha}(x) = +\infty\}$$

both sides of equality (2) are finite and, therefore, the integral

$$\int_0^\delta \frac{\mu(B(x,t))}{t^{n-\alpha+1}}dt \tag{3}$$

is also finite outside the set $S = S_1 \bigcup S_2$. From the finiteness of the integral (3) it follows that

$$\varliminf_{t\to 0} \frac{\mu(B(x,t))}{t^{n-\alpha}} = 0$$

for $x \in \mathbb{R}^n \setminus S$. Therefore, equality (2) has the following form

$$\int_0^\delta \frac{d\mu(B(x,t))}{t^{n-\alpha}}dt = \frac{\mu(B(x,\delta))}{\delta^{n-\alpha}} + \int_0^\delta \frac{\mu(B(x,t))}{t^{n-\alpha+1}}, \quad x \in \mathbb{R}^n \setminus S. \tag{4}$$

In [4], it was proven that $\hat{H}_{n-\alpha}(S_1) = 0$, and this means $C_{n-\alpha}(S_1) = 0$ (see Lemma 1). It is known that the set S_2 also has zero $C_{n-\alpha}$-capacity (see, for example, [2, p. 222]).

Since $C_{n-\alpha}(S) = 0$, $(S = S_1 \bigcup S_2)$, then for any $\varepsilon > 0$, according to the first part of the lemma, there exists an open set G_ε, $S \subset G_\varepsilon$, with capacity $C_{n-\alpha}(G_\varepsilon) < \varepsilon$ such that the left-hand side of equality (4), and, hence, the right-hand side as well, uniformly converges to zero as $\delta \to 0$ on the set $\mathbb{R}^n \setminus G_\varepsilon$. Thus, from the non-negativity of the terms on the right-hand side of equality (4), we obtain that the family of functions $\frac{\mu(B(x,t))}{t^{n-\alpha}}$ uniformly converges to zero as $t \to 0$ on the set $\mathbb{R}^n \setminus G_\varepsilon$. The lemma is proved. □

3. MAIN RESULTS

Using Lemmas 1 and 2 for any non-integer number $p:\ 0 < p < 2$, similarly to the proof of [6, Theorem 1.4], one can prove that if $u(x)$ is a subharmonic function in the domain $D \subset \mathbb{R}^n$, then for any $\varepsilon > 0$, there exists an open set $G_{p,\varepsilon} \subset D$ with capacity $C_{n-2+p}(G_{p,\varepsilon}) < \varepsilon$ such that $u(x)$ belongs to the class C^p on compact subsets of the difference $D \backslash G_{p,\varepsilon}$. Therefore, combining all the above formulated results, we obtain the following general theorem.

Theorem 1. *Let $u(x)$ be a subharmonic function in the domain $D \subset \mathbb{R}^n$. Then, for any number $p:\ 0 \le p \le 2$ and for any $\varepsilon > 0$, there exists an open set $G_{p,\varepsilon} \subset D$ with capacity $C_{n-2+p}(G_{p,\varepsilon}) < \varepsilon$ (for $p = 2$ with Lebesgue measure $m(G_{2,\varepsilon}) < \varepsilon$) such that $u(x)$ belongs to the class C^p on compact subsets of the difference $D \setminus G_{p,\varepsilon}$.*

Proof. We carry out the proof as in the proof of the main theorem in [5] for $n = 2$, $0 < p < 2$. The case $n > 2$ is proved similarly. We will also study the differentiability of the potential

$$u(x) = \int \ln|x-y| d\mu(y)$$

for a finite Borel measure μ concentrated in the unit circle $B = \left\{x \in \mathbb{R}^2 : |x| < 1\right\}$, supp$\mu \subset B$.

For any $\varepsilon > 0$ and $p : 0 < p < 2$, as $G_{p,\varepsilon} \subset B$, we take an open set with capacity $C_p(G_{p,\varepsilon}) < \varepsilon$ such that for some monotonically non-decreasing function $\gamma(t) > 0$, $t \in (0, +\infty)$, $\gamma(t) \to 0$, the following inequalities hold as $t \to 0$,

a) $\mu(B(x,t)) \le \gamma(t) \cdot t^p$, $x \in \overline{B} \setminus G_{p,\varepsilon}$,

b) $\int_{|x-y|<t} \frac{d\mu(y)}{|x-y|^p} \le \gamma(t)$, $x \in \overline{B} \setminus G_{p,\varepsilon}$.

The existence of a set $G_{p,\varepsilon}$ satisfying conditions a) and b) follows from Lemma 2.

Case $0 < p < 1$. On the compact set $\overline{B} \setminus G_{p,\varepsilon}$, we estimate the difference

$$R(x,h) = u(x+h) - u(x), \quad x, x+h \in \overline{B} \setminus G_{p,\varepsilon}$$

i.e.,

$$R(x,h) = \int\limits_{|x-y|<2|h|} [\ln|x-y+h| - \ln|x-y|]\, d\mu(y)$$

$$+ \int\limits_{|x-y|\geq 2|h|} [\ln|x-y+h| - \ln|x-y|]\, d\mu(y) = I_1(x,h) + I_2(x,h).$$

Hence, we have

$$I_1(x,h) = \int\limits_{|x-y|<2|h|} [\ln|x-y+h| - \ln|x-y|]\, d\mu(y)$$

$$\leq \int\limits_{|x-y|<2|h|} \ln\frac{|x-y+h|}{|x-y|} d\mu(y) \leq \int\limits_{|x-y|<2|h|} \ln(1 + \frac{|h|}{|x-y|}) d\mu(y).$$

According to Lemma 2 from [4], the latter can be rewritten as follows

$$\int\limits_0^{2|h|} \ln(1+\frac{|h|}{t}) d\mu(B(x,t)) = \ln\frac{3}{2}\cdot\mu(B(x,2|h|)) + \int\limits_0^{2|h|} \frac{|h|}{t+|h|}\frac{\mu(B(x,t))}{t} dt$$

$$\leq 2\gamma(2|h|)|h|^p + \int\limits_0^{2|h|} (\frac{\mu(B(x,t)}{t^p}) t^{p-1} dt \leq \frac{4}{p}\gamma(2|h|)|h|^p = \alpha_1'(h)|h|^p \tag{5}$$

and $\alpha_1'(h) \to 0$ as $h \to 0$. In a similar way, it can be shown that

$$-I_1(x,h) \leq \alpha_1''(h)|h|^p, \tag{6}$$

where $\alpha_1''(h) \to 0$ at $h \to 0$. Combining inequalities (5) and (6), we obtain

$$|I_1(x,h)| \leq \alpha_1(h)|h|^p, \quad x,\, x+h \in \overline{B}\backslash G_{p,\varepsilon},$$

where $\alpha_1(h) = \max\{\alpha_1'(h), \alpha_1''(h)\}$.

Next, we estimate the value of $I_2(x,h)$. To do this, we use the following inequality for the kernel

$$|\ln|x-y+h| - \ln|x-y|| \leq C\frac{|h|}{|x-y|},$$

$x, y, h \in \mathbb{R}^n$, $|x-y| \geq 2|h|$, C is a constant.

For sufficiently small $|h|$, we have

$$|I_2(x,h)| \leq C|h| \int\limits_{|x-y|\geq 2|h|} \frac{d\mu(y)}{|x-y|} = C|h| \int\limits_{2|h|}^{+\infty} \frac{d\mu(B(x,t))}{t}$$

$$\leq C|h| \int\limits_1^{+\infty} \frac{\mu(B(x,t)}{t^2} dt + C|h|^p|h|^{1-p} \int\limits_{2\sqrt{|h|}}^{1} \frac{\gamma(t)}{t^{2-p}} dt$$

$$+ C|h|^p|h|^{1-p} \int\limits_{2|h|}^{2\sqrt{|h|}} \frac{\gamma(t)}{t^{2-p}} dt \leq C\mu(B(0,2))|h|$$

$$+ \frac{C}{1-p}|h|^p|h|^{1-p} \int\limits_{2\sqrt{h}}^{1} \frac{1}{t^{1-p}} d\gamma(t) + \frac{C}{1-p}|h|^p|h|^{1-p} \int\limits_{2|h|}^{2\sqrt{|h|}} \frac{1}{t^{1-p}} d\gamma(t)$$

$$\leq C\mu(B(0,2))|h| + \frac{C}{1-p}\sqrt{|h|^{1-p}}\left(\gamma(1) - \gamma\left(2\sqrt{|h|}\right)\right)|h|^p$$

$$+ \frac{C}{1-p}(\gamma\left(2\sqrt{|h|}\right) - \gamma(2|h|))|h|^p = \alpha_2(h)|h|^p,$$

where $\alpha_2(h) \to 0$ as $h \to 0$. As a result, we obtain the estimate

$$|R(x,h)| \leq \alpha(h)|h|^p,\ x,\ x+h \in \overline{B} \setminus G_{p,\varepsilon},$$

where $\alpha(h) \to 0$ as $h \to 0$.

Case $1 \leq p < 2$. Next, on the compact $\overline{B} \setminus G_\varepsilon$, we estimate the difference

$$R_j(x,h) = u^{(j)}(x+h) - \sum_{|j+l|\leq 1} \frac{u^{(j+l)}(x)}{l!}h^l, \quad |j| \leq 1.$$

Let's consider the case of $j = 0$, then

$$R_0(x,h) = \int\limits_{|x-y|<2|h|} [\ln|x-y+h| - \ln|x-y|]\, d\mu(y) - \sum_{i=1}^{2} h_i \int\limits_{|x-y|<2|h|} \frac{\partial}{\partial x_i}\ln|x-y|d\mu(y)$$

$$+ \int\limits_{|x-y|\geq 2|h|} \left[\ln|x-y+h| - \ln|x-y| - \sum_{i=1}^{2} h_i\frac{\partial}{\partial x_i}\ln|x-y|\right] d\mu(y)$$

$$= I_1(x,h) - I_2(x,h) + I_3(x,h).$$

Here the value $I_1(x,h)$ is estimated in exactly the same way as in the case of $p < 1$.

The value $I_2(x,h)$ is estimated using condition b):

$$|I_2(x,h)| \leq 2|h| \int\limits_{|x-y|<2|h|} \frac{d\mu(y)}{|x-y|} \leq 2|h|^p \int\limits_{|x-y|<2|h|} \frac{d\mu(y)}{|x-y|^p} \leq \alpha_2(h)|h|^p,$$

where $\alpha_2(h) \to 0$ as $h \to 0$.

It remains to estimate $I_3(x,h)$. We use the inequality for the kernel

$$\left|\ln|x-y+h| - \ln|x-y| - \sum_{i=1}^{2} h_i\frac{\partial}{\partial x_i}\ln|x-y|\right| \leq C\frac{|h|^2}{|x-y|^2},$$

$x, y, h \in \mathbb{R}^n$, $|x-y| \geq 2|h|$, C is constant.

So, we have

$$|I_3(x,h)| \leq C|h|^2 \int\limits_{|x-y|\geq 2|h|} \frac{d\mu(y)}{|x-y|^2} \leq C|h|^2 \int\limits_{2|h|}^{\infty} \frac{d\mu(B(x,t))}{t^2}.$$

Integrating by parts of the last integral and using condition a), we obtain

$$|I_3(x,h)| \leq \alpha_3(h)|h|,$$

$\alpha_3(h) \to 0$ as $h \to 0$. As a result, we obtain the estimate

$$|R_0(x,h)| \leq \alpha(h)|h|, \quad x, x+h \in \overline{B} \setminus G_\varepsilon, \quad \alpha(h) \to 0 \quad \text{as} \quad h \to 0.$$

The differences $R_j(x,h)$, $|j| = 1$ are estimated in exactly the same way. Thus, potential $u(x)$ belongs to class $C^p(\overline{B} \setminus G_\varepsilon)$. □

Next, let us give definitions of classes $C^k(E)$ and $Lip_k(E)$ (see [8, 26, 27]). We say that a function $u(x)$ defined on an arbitrary closed set $E \subset \mathbb{R}^n$ belongs to the class $C^k(E)$, $k \geq 0$, if there exist functions $u^{(j)}$, $|j| \leq k$ defined on E such that $u^{(0)} = u$ and

$$|u^{(j)}(x+h) - \sum_{|j+l|\leq k} \frac{u^{(j+l)}(x)}{l!} h^l| \leq \gamma(h)|h|^{k-|j|},$$

where $x,\ x+h \in E$, $0 \leq |j| \leq k$, $\gamma(h) \to 0$ and $h \to 0$.

A function $u(x)$ defined on E belongs to the class $Lip_k(E)$, $k > 0$, if there exist functions $u^{(j)}$, $|j| < k$ defined on E such that $u^{(0)} = u$ and

$$|u^{(j)}(x+h) - \sum_{|j+l|<k} \frac{u^{(j+l)}(x)}{l!} h^l| \leq M|h|^{k-|j|},$$

where $x,\ x+h \in E$, $|j| < k$. Everywhere $j = (j_1, j_2, \dots, j_n)$ and $l = (l_1, l_2, \dots, l_n)$ are multi-indices, $|j| = j_1 + j_2 + \dots + j_n$, $l! = l_1! l_2! \dots l_n!$, $h^l = h_1^{l_1} h_2^{l_2} \dots h_n^{l_n}$.

Note that in the case of $E = \mathbb{R}^n$, the space $C^k(\mathbb{R}^n)$ has a simple differential property: it consists of functions that have continuous partial derivatives up to the mth order. The following theorem by Uitney [26] (see also [27]) on the extension of differentiable functions establishes a connection between $C^k(E)$ and $C^k(\mathbb{R}^n)$: let $\subset \mathbb{R}^n$ be a closed set and $u(x) \in C^k(E)$. Then, $u(x)$ extends to the whole space $\mathbb{R}^n$ as a function of class $C^k(\mathbb{R}^n)$, i.e., there exists a function $F(x) \in C^k(\mathbb{R}^n)$ such that $F(x) \equiv u(x)$.

A sufficient condition was established for the Riesz potential $U^\mu_{n-\alpha}(x)$ to belong to the Lipschitz class in [29]. It was shown that the Riesz potential $U^\mu_{n-\alpha}(x)$ belongs to the Lipschitz class $Lip_\beta(\mathbb{R}^n)$ of order β if for any real numbers α and β, with $0 < \alpha < n$, $0 < \beta < 1$ and for μ, the finite Borel measure with compact support on $\mathbb{R}^n$, the following inequality is satisfied

$$\mu(B(x,r)) \leq \text{const} \cdot r^{n+\beta-\alpha}, \tag{7}$$

where $B(x,r)$ is a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$.

Later, Wallin [28] proved that this condition is not only sufficient but also a necessary condition.

Note that for $\beta = 1$, the condition (7) is generally not sufficient. Later on, we will construct an example of a potential for which condition (7) is satisfied, but the potential itself does not belong to the class $Lip_1(\mathbb{R}^n)$. Here we present our second theorem, which complements the results given in [29].

Theorem 2. *Let $1 \leq \alpha < n$ and μ be positive finite Borel measures with compact support in $\mathbb{R}^n$ such that the inequality*

$$\mu(B(x,r)) \leq \text{const} \cdot r^{n+1-\alpha} \tag{8}$$

is satisfied for all $x \in \mathbb{R}^n$ and all $r > 0$. Then, the Riesz potential $U^\mu_{n-\alpha}(x)$ belongs to the Zygmund class $\Lambda_1(\mathbb{R}^n)$, i.e., for all $x, h \in \mathbb{R}^n$

$$\left|U^\mu_{n-\alpha}(x+h) - 2U^\mu_{n-\alpha}(x) + U^\mu_{n-\alpha}(x-h)\right| \leq C|h|,$$

where C is a constant number.

For examples of functions satisfying condition (8) see [8, p. 307].

Proof. Let the potential $U^\mu_{n-\alpha}(x)$, $1 \leq \alpha < n$ and μ be a finite Borel measure with compact support, which has the following condition

$$\mu(B(x,r)) \leq c\, r^{(n+1-\alpha)},$$

for any $x \in \mathbb{R}^n$ and $r > 0$, where c is a constant. We estimate the following difference

$$R(x,h) = U^\mu_{n-\alpha}(x+h) - 2U^\mu_{n-\alpha}(x) + U^\mu_{n-\alpha}(x)$$

for all $x, h \in \mathbb{R}^n$, i.e.,

$$R(x,h) = \int\limits_{|x-y|<2|h|} \frac{d\mu(y)}{|x-y+h|^{n-\alpha}} - 2\int\limits_{|x-y|<2|h|} \frac{d\mu(y)}{|x-y|^{n-\alpha}} + \int\limits_{|x-y|<2|h|} \frac{d\mu(y)}{|x-y-h|^{n-\alpha}}$$

$$+\int\limits_{|x-y|\ge 2|h|}\left[\frac{1}{|x-y+h|^{n-\alpha}}-2\frac{1}{|x-y|^{n-\alpha}}+\frac{1}{|x-y-h|^{n-\alpha}}\right]d\mu(y)$$
$$=I_1(x,h)+I_2(x,h)+I_3(x,h)+I_4(x,h).$$

We have an assessment of

$$I_1(x,h)=\int\limits_{|x-y|<2|h|}\frac{d\mu(y)}{|x-y+h|^{n-\alpha}}\le\int\limits_{|x-y+h|<3|h|}\frac{d\mu(y)}{|x-y+h|^{n-\alpha}}$$
$$=\int\limits_0^{3|h|}\frac{d\mu(B(x+h,t))}{t^{n-\alpha}}=\frac{\mu(B(x+h,3|h|))}{(3|h|)^{n+1-\alpha}}3|h|$$
$$+(n-\alpha)\int\limits_0^{3|h|}\frac{\mu(B(x+h,t))}{t^{n+1-\alpha}}dt\le 3c(n+1-\alpha)|h|=C_1|h|.$$

So, $0\le I_1(x,h)\le C_1|h|$, $\forall x,h\in\mathbb{R}^n$.

The values $I_2(x,h)$ and $I_3(x,h)$ are estimated in exactly the same way:

$0\le I_2(x,h)\le C_2|h|$, $\forall x,h\in\mathbb{R}^n$, where $C_2=2c(n+1-\alpha)$ and

$0\le I_3(x,h)\le C_3|h|$, $\forall x,h\in\mathbb{R}^n$, where $C_3=3c\,(n+1-\alpha)$.

Now it remains to evaluate $I_4(x,h)$. We will use the inequality for the kernel

$$\left|\frac{1}{|x-y+h|^{n-\alpha}}-2\frac{1}{|x-y|^{n-\alpha}}+\frac{1}{|x-y-h|^{n-\alpha}}\right|\le M\frac{|h|^2}{|x-y|^{n+2-\alpha}},$$

$x,y,h\in\mathbb{R}^n$, $|x-y|\ge 2|h|$, M is constant.

We have an estimate

$$|I_4|\le M|h|^2\int\limits_{|x-y|\ge 2|h|}\frac{d\mu(y)}{|x-y|^{n+2-\alpha}}=M|h|^2\int\limits_{2|h|}^{+\infty}\frac{d\mu(B(x,t))}{t^{n+2-\alpha}}$$
$$\le M(n+2-\alpha)|h|^2\int\limits_{2|h|}^{+\infty}\frac{\mu(B(x,t))}{t^{n+1-\alpha}}t^{-2}dt\le Mc(n+2-\alpha)|h|^2\int\limits_{2|h|}^{+\infty}\frac{dt}{t^2}=C_4|h|.$$

Combining all the estimates, we end up with the relation

$$|R(x,h)|\le C\,|h|,\forall x,h\in\mathbb{R}^n.$$

Thus, the potential $U^{\mu}_{n-\alpha}(x)$ belongs to the Zygmund class. The theorem is proved. □

An analogue of Theorem 2 is also true in the case of a logarithmic potential.

Theorem 3. *Let μ be a positive finite Borel measure with compact support μ in $\mathbb{R}^2$ such that the inequality*

$$\mu(B(x,r))\le c\,r \tag{9}$$

is satisfied for $\forall x\in\mathbb{R}^2$ and all C. Then, the logarithmic potential

$$U^{\mu}_0(x)=\int\ln|x-y|d\mu(y)$$

belongs to the Zygmund class $\Lambda_1(\mathbb{R}^2)$.

In fact, this theorem was proven in [13, Lemma 1].

Next, we give an example of a logarithmic potential for which conditions (9) are satisfied, but the potential itself does not belong to the Lipschitz class $Lip_1(\mathbb{R}^2)$.

Example. Let $E \subset \mathbb{C} \cong \mathbb{R}^2$ be a Cantor set such that the Hausdorff measure is finite, i.e., $0 \leq H_1(E) \leq +\infty$ and the analytic capacity is zero (see [30]). Consider the following logarithmic potential

$$u(z) = \int \ln|z-w| d\mu(w),$$

where $z = x_1 + ix_2$, $w = y_1 + iy_2$, $d\mu = dH_1$, i.e., supp$\mu \subseteq E$. It is clear that $u(z) \notin Lip_1(\mathbb{C})$. Because the function

$$\frac{\partial u(z)}{\partial z} = \int \frac{d\mu(w)}{z-w}$$

is holomorphic in the domain $\mathbb{C} \setminus E$ and is not bounded.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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