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ON DYNAMICS OF A NON-VOLTERRA QUADRATIC OPERATOR

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Novolterra kvadratik operator dinamikasi

Mazkur maqolada ikki o'lchamli simpleksda aniqlangan novolterra kvadratik stoxastik operatorining dinamikasi o'rganilgan. Ushbu operatorlar uchun davriy va qo'zg'almas nuqtalar to'plamlari topilgan hamda trayektoriyalarning limit nuqtalari to'plamlari tavsiflangan.

Kalit so'zlar: kvadratik stoxastik operator; Volterra operatori; novolterra operatori; trayektoriya.

Динамика невольтерровского квадратичного оператора

В статье изучена динамика невольтерровского квадратичного оператора в двумерном симплексе. Для такого оператора найдены множества периодических и неподвижных точек, а также описаны множества предельных точек траекторий.

Ключевые слова: квадратичный стохастический оператор; вольтерровский оператор; невольтерровский оператор; траектория.

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Keywords: quadratic stochastic operator; Volterra operator; non-Volterra operator; trajectory.

Introduction

The evolution of a population can be studied by a dynamical system of a quadratic stochastic operator [15]. Such evolution operators frequently arise in many models of mathematical genetics, namely theory of heredity (see e.g. [1, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

Let $E = \{1, \dots, m\}$ be a finite set and the set of all probability distributions on the set E

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}$$

the $(m-1)$ -dimensional simplex. A *quadratic stochastic operator* (QSO) is a mapping $V : S^{m-1} \rightarrow S^{m-1}$ given by

$$V : x'_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad k \in E \quad (1)$$

and the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^m p_{ij,k} = 1, \quad i, j, k \in E. \quad (2)$$

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The trajectory $\{\mathbf{x}^{(n)}\}_{n \geq 0}$, of QSO V for an initial point $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{n+1}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

Denote by $\omega_V(\mathbf{x}^{(0)})$ the ω -limit set of the trajectory $\{\mathbf{x}^{(n)}\}_{n \geq 0}$.

The main problem in mathematical population genetics consists of the study of the asymptotical behaviour of the trajectories for a given QSO (see e.g. [15],[16]). In other words, the main task is the description of the set $\omega_V(\mathbf{x}^{(0)})$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO. This problem is an open problem even in two-dimensional case. This problem deeply studied for the Volterra QSOs (see [5, 6]).

Definition 1. A quadratic stochastic operator V is called *Volterra* if

$$p_{ij,k} = 0, \quad \text{for any } k \notin \{i, j\}, i, j, k = 1, \dots, m.$$

A QSO V is called *regular* if there is the limit $\lim_{n \rightarrow \infty} V^n(\mathbf{x})$ for any initial $\mathbf{x} \in S^{m-1}$.

Let the set $\partial S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_i = 0 \text{ for at least one } i \in E\}$ be the boundary of the simplex S^{m-1} ; the set $\text{int } S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_1 x_2 \dots x_m > 0\}$ the interior of S^{m-1} and $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \dots, \delta_{mi}) \in S^{m-1}$, $i = 1, \dots, m$, the vertices of the simplex S^{m-1} , where δ_{ij} is the Kronecker delta.

In the present paper we consider a non-Volterra quadratic stochastic operator which depend of parameter $\theta \in [0, 1]$. Namely we consider the following quadratic stochastic operator defined on the S^2

$$V : \begin{cases} x'_1 = (1 - \theta)x_1^2 + \theta x_2^2 + x_1 x_2 + x_1 x_3, \\ x'_2 = (1 - \theta)x_2^2 + \theta x_3^2 + x_1 x_2 + x_2 x_3, \\ x'_3 = (1 - \theta)x_3^2 + \theta x_1^2 + x_1 x_3 + x_2 x_3, \end{cases} \quad (3)$$

where $0 \leq \theta \leq 1$. Note that the operator (3) is a non-Volterra QSO.

We proved that the center of the simplex is a unique fixed point for this operator. Also therein we proved that if $\theta = 1$ then there is a unique periodic point with the period three. We showed that for any initial point except the periodic points when $\theta = 1$ the trajectory of the operator approaches to the center of the simplex.

The uniqueness of a fixed point

A point $\mathbf{x}^* \in S^{m-1}$ is called a *periodic* point of V if there exists an n so that $V^n(\mathbf{x}^*) = \mathbf{x}^*$. The smallest positive integer n satisfying the above is called the prime period or least period of the point \mathbf{x}^* . A period-one point is called a *fixed* point of V .

Denote the set of all fixed points by $\text{Fix}(V)$ and the set of all periodic points of (not necessarily the smallest) period n by $\text{Per}_n(V)$. Evidently that the set of all iterates of a periodic point form a periodic trajectory (orbit).

In this subsection we shall examine the type of the fixed point $\mathbf{x}^* \equiv \mathbf{x}^*(\theta) = (x_1^*, x_2^*, x_3^*)$.

Definition 2. [3] A fixed point \mathbf{x}^* is said to be an *attracting* fixed point for V if there is a neighborhood U of \mathbf{x}^* such that if $\mathbf{x} \in U$, then $V^n(\mathbf{x}) \in U$ for all $n \geq 0$, and in fact $V^n(\mathbf{x}) \rightarrow \mathbf{x}^*$ as $n \rightarrow \infty$.

Definition 3. [3] A fixed point \mathbf{x}^* is said to be an *repelling* fixed point for V if there is a deleted neighborhood U of \mathbf{x}^* such that such that if $\mathbf{x} \in U$, then $V^n(\mathbf{x}) \notin U$ for some $n > 0$.

This means that an orbit with an initial condition starting even very close to \mathbf{x}^* will eventually need to move away from \mathbf{x}^* . Note that the orbit doesn't have to go to infinity or anywhere in particular, it just has to move away from \mathbf{x}^* .

Let $DV(\mathbf{x}^*) = (\partial V_i / \partial x_j(\mathbf{x}^*))_{i,j=1}^m$ be the Jacobian of V at the point \mathbf{x}^* .

Definition 4. [3] A fixed point \mathbf{x}^* is called *hyperbolic* if its Jacobian $DV(\mathbf{x}^*)$ has no eigenvalues on the unit sphere. It maybe that a hyperbolic fixed point \mathbf{x}^* is attracting, repelling, or neither.

Theorem 1. Suppose that \mathbf{x}^* is a hyperbolic fixed point for V . Then \mathbf{x}^* is:

- (i) *attracting* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are inside the unit ball;
- (ii) *repelling* if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are outside the unit ball;
- (iii) *a saddle* otherwise.

Theorem 2. For the operator V the following statements are true:

- (i) if $\theta = 0$ then $\text{Fix}(V) = S^2$;

(ii) if $0 < \theta \leq 1$ then $\text{Fix}(V) = \{\mathbf{c}\}$;

(iii) if $\theta = 0$ then \mathbf{c} is a non-hyperbolic point and it is an attracting point when $0 < \theta \leq 1$.

(iv) if $\theta = 1$ then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \text{Per}_3(V)$.

Proof. (i) Obviously.

(ii) A fixed point of the operator (3) is a solution of the equation $V(\mathbf{x}) = \mathbf{x}$ or, in other words, of the system

$$\begin{cases} x_1 = (1 - \theta)x_1^2 + \theta x_2^2 + x_1x_2 + x_1x_3, \\ x_2 = (1 - \theta)x_2^2 + \theta x_3^2 + x_1x_2 + x_2x_3, \\ x_3 = (1 - \theta)x_3^2 + \theta x_1^2 + x_1x_3 + x_2x_3, \end{cases} \quad (4)$$

where $0 < \theta \leq 1$.

From the first and second equations of the system (4) we have

$$\begin{aligned} x_1 - x_2 &= (1 - \theta)(x_1^2 - x_2^2) + \theta(x_2^2 - x_3^2) + x_3(x_1 - x_2) \Rightarrow \\ (x_1 - x_2)(1 - (1 - \theta)(x_1 + x_2) - x_3) &= \theta(x_2 - x_3)(x_2 + x_3) \Rightarrow \\ (x_1 - x_2)(x_1 + x_2) &= (x_2 - x_3)(x_2 + x_3). \end{aligned}$$

Similarly from the second and third equations of the system (4) we obtain

$$(x_2 - x_3)(x_2 + x_3) = (x_3 - x_1)(x_2 + x_3).$$

Analogously from the third and first equations of the system (4) we obtain

$$(x_3 - x_1)(x_3 + x_1) = (x_1 - x_2)(x_1 + x_2).$$

Therefore the system (4) can be written in the following form

$$\begin{cases} (x_1 - x_2)(x_1 + x_2) = (x_2 - x_3)(x_2 + x_3), \\ (x_2 - x_3)(x_2 + x_3) = (x_3 - x_1)(x_2 + x_3), \\ (x_3 - x_1)(x_3 + x_1) = (x_1 - x_2)(x_1 + x_2). \end{cases} \quad (5)$$

Suppose that $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$ be a fixed point and $x_1^* \geq x_2^* \geq x_3^*$ then from the system (5) one has that

$$x_1^* \geq x_2^* \geq x_3^* \geq x_1^* \Rightarrow x_1^* = x_2^* = x_3^*.$$

As $x_1^* + x_2^* + x_3^* = 1$ it follows that $x_1^* = x_2^* = x_3^* = 1/3$. Hence we have that $\mathbf{x}^* = \mathbf{c}$ is the center of the simplex.

(iii) To find the type of a fixed point of the operator (3) we write as follows:

$$\begin{cases} x_1' = \theta x_2^2 - \theta x_1^2 + x_1 \\ x_2' = \theta x_1^2 + 2\theta x_1x_2 - 2\theta x_1 + (1 - 2\theta)x_2 + \theta, \end{cases} \quad (6)$$

where $(x_1, x_2) \in \{(x, y) : x, y \geq 0, 0 \leq x + y \leq 1\}$ and x_1, x_2 , are the first two coordinates of a point lying in the simplex S^2 .

$$\begin{aligned} \frac{\partial x_1'}{\partial x_1} &= 1 - 2\theta x_1, & \frac{\partial x_1'}{\partial x_2} &= 2\theta x_2 \\ \frac{\partial x_2'}{\partial x_1} &= 2\theta x_1 + 2\theta x_2 - 2\theta, & \frac{\partial x_2'}{\partial x_2} &= 2\theta x_1 + (1 - 2\theta) \end{aligned}$$

The Jacobi matrix of the (3) at the center has the form

$$J_V(\mathbf{c}) = \begin{pmatrix} \frac{3 - 2\theta}{3} & \frac{2\theta}{3} \\ -\frac{2\theta}{3} & \frac{3 - 4\theta}{3} \end{pmatrix}. \quad (7)$$

One has that the Jacobi matrix (7) has the eigenvalues

$$\mu_{1,2} = 1 - \theta \pm \frac{\theta}{\sqrt{3}}i$$

and $|\mu_{1,2}| < 1$ when $\theta \in (0, 1]$. It is clear that if $\theta = 0$ then $|\mu_{1,2}| = 1$.

Consequently we have that the center

$$\mathbf{c} \text{ has type } \begin{cases} \text{non-hyperbolic} & \text{if } \theta = 0, \\ \text{attracting} & \text{if } 0 < \theta \leq 1. \end{cases}$$

(iv) It is easy to see that $V^3(\mathbf{e}_1) = V^2(\mathbf{e}_3) = V(\mathbf{e}_2) = \mathbf{e}_1$.

The theorem 2 is proved. □

The ω - limit set

The problem of describing the ω - limit set of a trajectory is of great importance in the theory of dynamical systems.

Let $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \in S^2$ be the initial point and let $\{\mathbf{x}^{(n)}\}_{n \geq 0}$ be the trajectory of $\mathbf{x}^{(0)}$ under the action of the operator (3); that is,

$$\mathbf{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) = V(\mathbf{x}^{(n-1)}), \quad n = 1, 2, \dots$$

Let $\mathbf{c} = (1/3, 1/3, 1/3)$ be the center of the simplex S^2 .

Recall that a continuous function $\varphi : S^{m-1} \rightarrow \mathbb{R}$ is called a Lyapunov function for a QSO V if there exists the limit $\lim_{n \rightarrow \infty} \varphi(\mathbf{x}^{(n)})$ for all $\mathbf{x} \in S^{m-1}$.

Theorem 3. *Let $0 < \theta < 1$. Then for the non-Volterra QSO (3) the following assertions are true:*

(i) *the function $\varphi(\mathbf{x}) = |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|$ is a Lyapunov function;*

(ii) *the function $\psi(\mathbf{x}) = \max_{i \in \{1,2,3\}} |x_i - 1/3|$ is a Lyapunov function.*

Proof. Since for any $\mathbf{x} \in \partial S^2$ it follows that $V(\mathbf{x}) \in \text{int } S^2$ we without loss of generality we assume that $\mathbf{x} \in \text{int } S^2$.

(i) Let $0 < \theta < 1$. From (3) we have the following differences

$$\begin{aligned} x'_1 - x'_2 &= (1 - \theta)(x_1^2 - x_2^2) + \theta(x_2^2 - x_3^2) + x_3(x_1 - x_2) = (x_1 - x_2)(1 - \theta(x_1 + x_2)) + \theta(x_2 - x_3)(x_2 + x_3), \\ x'_2 - x'_3 &= (1 - \theta)(x_2^2 - x_3^2) + \theta(x_3^2 - x_1^2) + x_1(x_2 - x_3) = (x_2 - x_3)(1 - \theta(x_2 + x_3)) + \theta(x_3 - x_1)(x_1 + x_3), \\ x'_3 - x'_1 &= (1 - \theta)(x_3^2 - x_1^2) + \theta(x_1^2 - x_2^2) + x_2(x_3 - x_1) = (x_3 - x_1)(1 - \theta(x_1 + x_3)) + \theta(x_1 - x_2)(x_1 + x_2). \end{aligned}$$

Then using $0 < \theta < 1$ for any $\mathbf{x} \in \text{int } S^2$ one has that

$$\begin{aligned} |x'_1 - x'_2| &\leq |x_1 - x_2|(1 - \theta(x_1 + x_2)) + \theta|x_2 - x_3|(x_2 + x_3) \\ |x'_2 - x'_3| &\leq |x_2 - x_3|(1 - \theta(x_2 + x_3)) + \theta|x_3 - x_1|(x_1 + x_3) \\ |x'_3 - x'_1| &\leq |x_3 - x_1|(1 - \theta(x_1 + x_3)) + \theta|x_1 - x_2|(x_1 + x_2) \end{aligned} \tag{8}$$

But for any $\mathbf{x} \in \text{int } S^2 \setminus \{\mathbf{c}\}$ using $0 < \theta < 1$ from (8) we have

$$\begin{aligned} |x'_1 - x'_2| &< |x_1 - x_2|(1 - \theta(x_1 + x_2)) + \theta|x_2 - x_3|(x_2 + x_3) \\ |x'_2 - x'_3| &< |x_2 - x_3|(1 - \theta(x_2 + x_3)) + \theta|x_3 - x_1|(x_1 + x_3) \\ |x'_3 - x'_1| &< |x_3 - x_1|(1 - \theta(x_1 + x_3)) + \theta|x_1 - x_2|(x_1 + x_2) \end{aligned} \tag{9}$$

Consider the function $\varphi : S^2 \rightarrow \mathbb{R}$ defined as follows

$$\varphi(\mathbf{x}) = |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|, \quad \mathbf{x} \in S^2.$$

For any $\mathbf{x} \in \text{int } S^2$ from (9) we have $\varphi(V(\mathbf{x})) \leq \varphi(\mathbf{x})$. Thus $\varphi(\mathbf{x})$ is Lyapunov function for the operator V .

(ii) Let $0 < \theta < 1$ and $\mathbf{x} \in S^2$ then one has that

$$\begin{aligned} x'_1 - \frac{1}{3} &= x_1 - \frac{1}{3} + \theta(x_2 - x_1)(x_2 + x_1) = x_1 - \frac{1}{3} + \theta \left(x_2 - \frac{1}{3}\right)(x_2 + x_1) - \theta \left(x_1 - \frac{1}{3}\right)(x_2 + x_1) \\ &= \left(x_1 - \frac{1}{3}\right)(1 - \theta(x_2 + x_1)) + \theta \left(x_2 - \frac{1}{3}\right)(x_2 + x_1), \\ x'_2 - \frac{1}{3} &= x_2 - \frac{1}{3} + \theta(x_3 - x_2)(x_3 + x_2) = x_2 - \frac{1}{3} + \theta \left(x_3 - \frac{1}{3}\right)(x_3 + x_2) - \theta \left(x_2 - \frac{1}{3}\right)(x_3 + x_2) \\ &= \left(x_2 - \frac{1}{3}\right)(1 - \theta(x_3 + x_2)) + \theta \left(x_3 - \frac{1}{3}\right)(x_3 + x_2), \\ x'_3 - \frac{1}{3} &= x_3 - \frac{1}{3} + \theta(x_1 - x_3)(x_1 + x_3) = x_3 - \frac{1}{3} + \theta \left(x_1 - \frac{1}{3}\right)(x_1 + x_3) - \theta \left(x_3 - \frac{1}{3}\right)(x_1 + x_3) \\ &= \left(x_3 - \frac{1}{3}\right)(1 - \theta(x_1 + x_3)) + \theta \left(x_1 - \frac{1}{3}\right)(x_1 + x_3). \end{aligned} \quad (10)$$

Then using $0 < \theta < 1$ and $\mathbf{x}^{(n)} \in S^2$ from (10) we have

$$\begin{aligned} \left|x'_1 - \frac{1}{3}\right| &\leq \max \left\{ \left|x_1 - \frac{1}{3}\right|, \left|x_2 - \frac{1}{3}\right| \right\}, \\ \left|x'_2 - \frac{1}{3}\right| &\leq \max \left\{ \left|x_2 - \frac{1}{3}\right|, \left|x_3 - \frac{1}{3}\right| \right\}, \\ \left|x'_3 - \frac{1}{3}\right| &\leq \max \left\{ \left|x_3 - \frac{1}{3}\right|, \left|x_1 - \frac{1}{3}\right| \right\}. \end{aligned} \quad (11)$$

Consider the function $\psi : S^2 \rightarrow \mathbb{R}$ defined as follows

$$\psi(\mathbf{x}) = \max_{i \in \{1,2,3\}} \left| x_i - \frac{1}{3} \right|, \quad \mathbf{x} \in S^2.$$

For any $0 < \theta < 1$ and $\mathbf{x}^{(n)} \in S^2$ from (11) we have $\psi(V(\mathbf{x})) < \psi(\mathbf{x})$, that is the function $\psi(\mathbf{x})$ is Lyapunov function for the operator V .

The theorem 3 is proved. \square

Theorem 4. For the non-Volterra QSO (3) the following assertions are true:

(i) If $\theta = 0$ then the operator V is identity map;

(ii) If $0 < \theta < 1$ then $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{c}$ for any $\mathbf{x} \in S^2$;

(iii) If $\theta = 1$ then for any $\mathbf{x} \in S^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{c}\}$ we have $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{c}$.

Proof. (i) Obviously.

(ii) By Theorem 3 the function $\psi(\mathbf{x}) = \max_{i \in \{1,2,3\}} \left| x_i - \frac{1}{3} \right|$ is a Lyapunov function. It is clear that $\psi(\mathbf{x})$ is a continuous function and $\psi(\mathbf{x}) = 0$ iff $\mathbf{x} = \mathbf{c}$ in the simplex S^2 .

For any $0 < \theta < 1$ and for any initial $\mathbf{x}^{(0)} \in \text{int } S^2 \setminus \{\mathbf{c}\}$ from (11) we have

$$\psi(\mathbf{x}^{(n+1)}) < \psi(\mathbf{x}^{(n)}) < \dots < \psi(\mathbf{x}^{(0)}).$$

Therefore we have that the sequence $\{\psi(\mathbf{x}^{(n)})\}_{n \geq 0}$ is a strictly decreasing and bounded sequence. So it follows that there exists the limit $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) = 0$. Suppose the converse. Suppose that $\{\psi(\mathbf{x}^{(n)})\}$ has a limit, but $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) \neq 0$, i.e. $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) = \tau > 0$. Then there is a sequence $\{x_{j_k}^{(n)}\}$ such that the coordinates satisfy

$\psi(\mathbf{x})$ and it follows that $\{x_{j_k}^{(n)}\} \rightarrow \frac{1}{3} \pm \tau$ as $n \rightarrow \infty$. Therefore from $\{x_{j_k}^{(n)}\}$ we can choose the subsequences $\{x_{j_k}^{(n_i)}\}$ and $\{x_{j_k}^{(n_t)}\}$ such that $\{x_{j_k}^{(n_i)}\} \rightarrow \frac{1}{3} + \tau$ and $\{x_{j_k}^{(n_t)}\} \rightarrow \frac{1}{3} - \tau$ as $i, t \rightarrow \infty$. Then using $x_1 + x_2 + x_3 = 1$ from $\{x_{j_k}^{(n_i)}\} \rightarrow \frac{1}{3} + \tau$ it follows that $\{x_{j_m}^{(n_i)}\} + \{x_{j_l}^{(n_i)}\} \rightarrow \frac{2}{3} - \tau$ as $i \rightarrow \infty$. If we denote $\left| x_{j_k}^{(n_i)} - \frac{1}{3} \right| = \tau_{n_i}$, $\left| x_{j_m}^{(n_i)} - \frac{1}{3} \right| = \tau'_{n_i}$ and $\left| x_{j_l}^{(n_i)} - \frac{1}{3} \right| = \tau''_{n_i}$ then we have $\tau_{n_i} \rightarrow \tau$ and $\tau'_{n_i} + \tau''_{n_i} \rightarrow \tau$ as $n \rightarrow \infty$. We claim that from $\{x_{j_k}^{(n_i)}\} \rightarrow \frac{1}{3} + \tau$ it follows that $\{x_{j_k}^{(n_i)}\} = \frac{1}{3} + \tau_{n_i}$, $\{x_{j_m}^{(n_i)}\} = \frac{1}{3} - \tau'_{n_i}$ and $\{x_{j_l}^{(n_i)}\} = \frac{1}{3} - \tau''_{n_i}$, that is, we obtain that if $\{x_{j_k}^{(n_i)}\} - \frac{1}{3} \geq 0$ then $\{x_{j_m}^{(n_i)}\} - \frac{1}{3} \leq 0$ and $\{x_{j_l}^{(n_i)}\} - \frac{1}{3} \leq 0$.

Indeed, suppose on the contrary: (ii_a) if with the inequality $\{x_{j_k}^{(n_i)}\} - \frac{1}{3} \geq 0$ it hold $\{x_{j_m}^{(n_i)}\} - \frac{1}{3} \geq 0$ and $\{x_{j_l}^{(n_i)}\} - \frac{1}{3} \geq 0$ then we get $\{x_{j_k}^{(n_i)}\} + \{x_{j_m}^{(n_i)}\} + \{x_{j_l}^{(n_i)}\} \geq 1$. The last inequality is true only at the center \mathbf{c} of the S^2 , but this impossible by assumption $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) \neq 0$.

(ii_b) if with the inequality $\{x_{j_k}^{(n_i)}\} - \frac{1}{3} \geq 0$ for example say, it holds $\{x_{j_m}^{(n_i)}\} - \frac{1}{3} \geq 0$ and $\{x_{j_l}^{(n_i)}\} - \frac{1}{3} \leq 0$ then we have that $\{x_{j_k}^{(n_i)}\} + \{x_{j_m}^{(n_i)}\} = \frac{2}{3} + \tau_{n_i} + \tau'_{n_i}$ and using $\{x_{j_k}^{(n_i)}\} + \{x_{j_m}^{(n_i)}\} + \{x_{j_l}^{(n_i)}\} = 1$ we get $\{x_{j_l}^{(n_i)}\} = \frac{1}{3} - (\tau_{n_i} + \tau'_{n_i})$. Therefore it follows that $\left| \{x_{j_l}^{(n_i)}\} - \frac{1}{3} \right| = \tau_{n_i} + \tau'_{n_i}$. But the last is a contradiction to the fact that $\left| \{x_{j_k}^{(n_i)}\} - \frac{1}{3} \right|$ in the maximum.

Similarly can be showed the case $\{x_{j_k}^{(n_i)}\} - \frac{1}{3} \leq 0$, $\{x_{j_m}^{(n_i)}\} - \frac{1}{3} \leq 0$ and $\{x_{j_l}^{(n_i)}\} - \frac{1}{3} \geq 0$.

Another word we proved that if $\left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ in the maximum then one has that the coordinate $x_{j_k}^{(n)}$ lies one side from the point $\frac{1}{3}$ and the coordinates $x_{j_m}^{(n)}, x_{j_l}^{(n)}$ lie to the another side from the point $\frac{1}{3}$.

From $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) = \tau > 0$ it follows that there is the following limit

$$1 = \lim_{n \rightarrow \infty} \frac{\psi(\mathbf{x}^{(n+1)})}{\psi(\mathbf{x}^{(n)})} = \lim_{n \rightarrow \infty} \left| \frac{\left(x_{j_m}^{(n)} - \frac{1}{3} \right) \left(1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right) + \left(x_{j_l}^{(n)} - \frac{1}{3} \right) \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)})}{x_{j_k}^{(n)} - \frac{1}{3}} \right|. \tag{12}$$

We consider all possible cases:

a) Suppose that $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$. Then from (12) one has

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \left| \frac{\left(x_{j_m}^{(n)} - \frac{1}{3} \right) \left(1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right) + \left(x_{j_l}^{(n)} - \frac{1}{3} \right) \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)})}{x_{j_k}^{(n)} - \frac{1}{3}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x_{j_m}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \cdot \left(1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right) + \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \cdot \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right| \\ &\leq \lim_{n \rightarrow \infty} \left(\left| \frac{x_{j_m}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| \left(1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right) + \left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right) \\ &\leq \lim_{n \rightarrow \infty} \left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right|, \end{aligned}$$

where we have used the following assumption $\left| \frac{x_{j_m}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| < \left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right|$. Since $\left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| < 1$ we have

$$\lim_{n \rightarrow \infty} \left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| = 1. \quad (13)$$

Since $\left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ maximum it follows that the coordinate $x_{j_k}^{(n)}$ lies one side from the point $\frac{1}{3}$ and the coordinates $x_{j_m}^{(n)}, x_{j_l}^{(n)}$ lie to the another side. Otherwise we get the contradiction to the fact $\left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ is the maximum. Therefore using the $j_k \neq j_l$ we have that

$$\lim_{n \rightarrow \infty} \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} = -1.$$

From the last it follows that $\{x_{j_m}^{(n_i)}\} \rightarrow \frac{1}{3}$. Using it from the equation (12) we have

$$\begin{aligned} 1 &= \lim_{i \rightarrow \infty} \left| \left(\frac{x_{j_m}^{(n_i)} - \frac{1}{3}}{x_{j_k}^{(n_i)} - \frac{1}{3}} \right) (1 - \theta(x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)})) + \left(\frac{x_{j_l}^{(n_i)} - \frac{1}{3}}{x_{j_k}^{(n_i)} - \frac{1}{3}} \right) \theta(x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)}) \right| \\ &\leq \lim_{i \rightarrow \infty} \left| \frac{x_{j_m}^{(n_i)} - \frac{1}{3}}{x_{j_k}^{(n_i)} - \frac{1}{3}} \right| (1 - \theta(x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)})) + \left| \frac{x_{j_l}^{(n_i)} - \frac{1}{3}}{x_{j_k}^{(n_i)} - \frac{1}{3}} \right| \theta(x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)}) \\ &= \lim_{i \rightarrow \infty} (1 - \theta(x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)})). \end{aligned}$$

Using the last relation and $\theta > 0$ it follows that $x_{j_m}^{(n_i)} + x_{j_l}^{(n_i)} \rightarrow 0$ as $i \rightarrow \infty$. But this contradicts to the fact $\{x_{j_m}^{(n_i)}\} \rightarrow \frac{1}{3}$ as $i \rightarrow \infty$.

b) Suppose that $j_m = j_k$ and $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$. Then from (12) we have the following contradiction

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \left| \frac{\left(x_{j_k}^{(n)} - \frac{1}{3} \right) (1 - \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)})) + \left(x_{j_l}^{(n)} - \frac{1}{3} \right) \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)})}{x_{j_k}^{(n)} - \frac{1}{3}} \right| \\ &= \lim_{n \rightarrow \infty} \left| 1 - \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) + \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) \right| \\ &= \lim_{n \rightarrow \infty} \left| 1 + \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) \left(\frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} - 1 \right) \right| < 1, \end{aligned}$$

where we have used the following inequalities $0 \leq (x_{j_k}^{(n)} + x_{j_l}^{(n)}) < 1$ and $-1 < \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \leq 0$.

The case $j_l = j_k$ and $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ can be considered in a similar manner.

c) Assume that $j_m \neq j_k$, $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$. Then from the equality (12) it follows that

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \left| \frac{\left(x_{j_m}^{(n)} - \frac{1}{3} \right) (1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)})) + \left(x_{j_l}^{(n)} - \frac{1}{3} \right) \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)})}{x_{j_k}^{(n)} - \frac{1}{3}} \right| \\ &= \lim_{n \rightarrow \infty} \left| - (1 - \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)})) + \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \right| \\ &= \lim_{n \rightarrow \infty} \left| -1 + \theta(x_{j_m}^{(n)} + x_{j_l}^{(n)}) \left(\frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} + 1 \right) \right|. \end{aligned}$$

In this case it is hold $0 \leq (x_{j_k}^{(n)} + x_{j_l}^{(n)}) < 1$ and $-1 < \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \leq 0$. Consequently, we have

$$\lim_{n \rightarrow \infty} \left(\frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right) = -1, \quad \text{that is} \quad \lim_{n \rightarrow \infty} \left| \frac{x_{j_l}^{(n)} - \frac{1}{3}}{x_{j_k}^{(n)} - \frac{1}{3}} \right| = 1.$$

Thus we obtain that

$$\lim_{n \rightarrow \infty} \left| x_{j_k}^{(n)} - \frac{1}{3} \right| = \lim_{n \rightarrow \infty} \left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \lim_{n \rightarrow \infty} \left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \tau$$

and it contradicts to $\lim_{n \rightarrow \infty} (x_{j_k}^{(n)} + x_{j_l}^{(n)} + x_{j_m}^{(n)}) = 1$.

Similar result can be proved for the case $j_l \neq j_k$, $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| < \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$.

d) Suppose that $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and j_k, j_m, j_l are different coordinates. Then in this case we have that

$$\lim_{n \rightarrow \infty} \left| x_{j_k}^{(n)} - \frac{1}{3} \right| = \lim_{n \rightarrow \infty} \left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \lim_{n \rightarrow \infty} \left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \tau$$

and it contradicts to $\lim_{n \rightarrow \infty} (x_{j_k}^{(n)} + x_{j_l}^{(n)} + x_{j_m}^{(n)}) = 1$.

e) Suppose that $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$, $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $j_k = j_m$. In this case from (12) one has the following contradiction

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \left| \frac{\left(x_{j_k}^{(n)} - \frac{1}{3} \right) (1 - \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)})) + \left(x_{j_l}^{(n)} - \frac{1}{3} \right) \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)})}{x_{j_k}^{(n)} - \frac{1}{3}} \right| = \\ &= \lim_{n \rightarrow \infty} \left| 1 - \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) - \theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) \right| = \lim_{n \rightarrow \infty} \left| 1 - 2\theta(x_{j_k}^{(n)} + x_{j_l}^{(n)}) \right| < 1, \end{aligned}$$

where we have used $0 \leq (x_{j_k}^{(n)} + x_{j_l}^{(n)}) < 1$.

The case $\left| x_{j_m}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$, $\left| x_{j_l}^{(n)} - \frac{1}{3} \right| = \left| x_{j_k}^{(n)} - \frac{1}{3} \right|$ and $j_k = j_l$ can be considered in a similar manner.

Thus we have that $\tau = 0$ and it follows that $\lim_{n \rightarrow \infty} \psi(\mathbf{x}^{(n)}) = 0$, that is, $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{c}$.

(iii) Due to Theorem 2 in the case $\theta = 1$ there is a unique periodic trajectory $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, that is

$$V^3(\mathbf{e}_1) = V^2(\mathbf{e}_3) = V(\mathbf{e}_2) = \mathbf{e}_1.$$

If $\theta = 1$ and for any $\mathbf{x} \in S^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{c}\}$ we have $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{c}$. This can be proved as in the previous case. The theorem 4 is proved. \square

References

1. Bernstein S. The solution of a mathematical problem related to the theory of heredity. *Uchn. Zapiski. NI Kaf. Ukr. Otd. Mat.*, 1, 1924, pp. 83–115.
2. Blath J., Jamilov (Zhamilov) U. U., Scheutzow M. (G, μ) -quadratic stochastic operators. *J. Difference Equ. Appl.*, 20(8), 2014, pp. 1258–1267.
3. Devaney R. L. An introduction to chaotic dynamical systems. Studies in Nonlinearity, Westview Press, Boulder, CO, 2003, reprint of the second (1989) edition.
4. Ganikhodjaev N. N., Ganikhodjaev R. N., Jamilov U. U. Quadratic stochastic operators and zero-sum game dynamics. *Ergod. Th. and Dynam. Sys.*, 35(5), 2015, pp. 1443–1473.
5. R. N. Ganikhodzhaev Quadratic stochastic operators, Lyapunov functions and tournaments. *Sb. Math.* 76(2), 1993, pp. 489–506.
6. R. N. Ganikhodzhaev Map of fixed points and Lyapunov functions for one class of discrete dynamical systems. *Math. Notes* 56(5), 1994, pp. 1125–1131.
7. R. Ganikhodzhaev, F. Mukhamedov, U. Rozikov Quadratic stochastic operators and processes: results and open problems. *Infin. Dimens. Anal. Quan. Probab. Relat. Top.* 14(2), 2011, pp. 279–335.
8. U. U. Jamilov Quadratic stochastic operators corresponding to graphs. *Lobachevskii J. Math.* 34(2), 2013, pp. 148–151.
9. U. U. Jamilov On a family of strictly non-volterra quadratic stochastic operators. *Jour. Phys. Conf. Ser.* 697, 2016, 012013.
10. U. U. Jamilov, M. Ladra Non-ergodicity of uniform quadratic stochastic operators. *Qual. Theory Dyn. Syst.* 15(1), 2016, pp. 257–271.
11. U. U. Jamilov, Kh.O. Khudoyberdiev, M. Ladra Quadratic operators corresponding to permutations. *Stochas. Anal. Appl.* 38(5), 2020, pp. 929–938.
12. U. U. Jamilov, M. Scheutzow, M. Wilke-Berenguer On the random dynamics of Volterra quadratic operators. *Ergodic Theory Dynam. Systems.* 37(1), 2017, pp. 228–243.
13. H. Kesten Quadratic transformations: A model for population growth. I. *Advances in Appl. Probability* 2, 1970, pp. 1–82.
14. A. Yu. Khamraev On the dynamics of a quasi strictly non-Volterra quadratic stochastic operator. *Ukrainian Math. J.* 71(8), 2020, pp. 1273–1281.
15. Lyubich Y. I. Mathematical structures in population genetics. vol. 22 of Biomathematics, Springer-Verlag, Berlin, 1992.
16. Mukhamedov F., Ganikhodjaev N. Quantum Quadratic Operators and Processes, Springer, Berlin, 2015.

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