Regularity of a non-Volterra quadratic stochastic operator

Rajabov Suyunjon

V.I. Romanovskiy Institute of Mathematics

Uzbekistan Academy of Sciences

Tashkent, Uzbekistan

`rajabovs90@mail.ru`

Abstract

The present article deals with dynamics of a non-Volterra quadratic stochastic operator defined on the two-dimensional simplex. We showed that it has at least four fixed points and we found the types of all fixed points. Also we proved that for any values the parameters the trajectory of an arbitrary initial point approaches to a fixed point.

Keywords: quadratic stochastic operator; Volterra operator; non-Volterra operator; regularity.
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1. Introduction

The evolution of a population can be studied by a dynamical system of a quadratic stochastic operator [1]. Such evolution operators frequently arise in many models of mathematical genetics, namely theory of heredity (see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]).

Let $E_m = \{1, \ldots, m\}$ be a finite set. The set of all probability distributions on E_m is

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \ x_i \geq 0, \text{ for any } i \text{ and } \sum_{i=1}^{m} x_i = 1 \right\},$$

the $(m-1)$-dimensional simplex.

A *quadratic stochastic operator* (QSO) is a mapping $V : S^{m-1} \to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$V : x_k' = \sum_{i,j \in E_m} p_{ij,k} x_i x_j, \quad \forall k \in E_m \tag{1}$$

and the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k \in E_m}^{m} p_{ij,k} = 1, \quad \text{for all} \quad i, j, k \in E_m. \tag{2}$$

For an initial point $\mathbf{x}^{(0)} \in S^{m-1}$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\ldots}$ of $\mathbf{x}^{(0)}$ be of under action of V is defined by

$$\mathbf{x}^{(n+1)} = V\big(\mathbf{x}^{(n)}\big) = V^{n+1}\big(\mathbf{x}^{(0)}\big), \quad n = 0, 1, 2, \ldots$$

Denote by $\omega_V\big(\mathbf{x}^{(0)}\big)$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\ldots}$.

One of the main problems in mathematical biology consists in the study of the asymptotical behaviour of the trajectories for a given QSO (see e.g. [1]). In other words, the main task is the description of the set $\omega_V\big(\mathbf{x}^{(0)}\big)$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO. This problem is an open problem even the in two-dimensional case.

This problem was solved for the *Volterra* QSOs defined by (1), (2) and by the additional assumption

$$p_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}, \quad i, j, k \in E_m. \tag{3}$$

The biological treatment of conditions (3) is rather precise: the offspring repeats the genotype of one of its parents.

In [7, **?**] the theory of Volterra QSOs was developed using the theory of the Lyapunov functions and tournaments. But in dynamical systems generated by non-Volterra QSOs, many questions remain open and there seems to be no general theory available. To the best of our knowledge, there are few papers devoted to such operators. For a recent review on the theory of quadratic stochastic operators see [8].

In the present paper, we consider discrete-time dynamical systems generated by a non-Volterra quadratic stochastic operator. The article is organized as follows. In section 2 we provide necessary definition and notations from theory of QSOs. Also there in we gave the form of the non-Volterra QSO (4), which will be investigated in the next sections. In section 3 we found all fixed points of the QSO (4) and define their types. In section 4 for the QSO (4) we show that for an initial point the set of limit points is singleton.

2. Preliminaries

Let V be a quadratic stochastic operator. A point $\mathbf{x} \in S^{m-1}$ is called a *periodic* point of V if there exists an n such that $V^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer n satisfying the above equation is called the *prime period* or *least period* of the point $\mathbf{x}$. Denote the set of all periodic points (not necessarily prime) of period n by $\mathrm{Per}_n(V)$. A period-one point is called a *fixed* point of V and we denote the set of all fixed by $\mathrm{Fix}(V)$.

Let $DV(\mathbf{x}^*) = \left(\partial V_i / \partial x_j(\mathbf{x}^*) \right)_{i,j=1}^m$ be the Jacobi matrix of V at the point $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is called *hyperbolic* if its Jacobi matrix $DV(\mathbf{x}^*)$ has no eigenvalues 1 in absolute value. A hyperbolic fixed point $\mathbf{x}^*$ is called *attracting* (resp. *repelling*) if all the eigenvalues of the Jacobi matrix $DV(\mathbf{x}^*)$ are less (resp. greater) than 1 in absolute value; it is called a *saddle* if some of the eigenvalues of $DV(\mathbf{x}^*)$ are less than 1 in absolute value and other eigenvalues are greater than 1 in absolute value (see [18]).

A continuous function $\varphi \colon S^{m-1} \to \mathbb{R}$ is called a *Lyapunov function* for an operator V if there is the limit $\lim\limits_{n \to \infty} \varphi\left(\mathbf{x}^{(n)} \right)$ for all $\mathbf{x} \in S^{m-1}$.

A QSO V is called *regular* if there is the limit $\lim\limits_{n \to \infty} V^n(\mathbf{x})$ for any initial $\mathbf{x} \in S^{m-1}$.

A QSO V is said to be *ergodic* if the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(\mathbf{x})$$

exists for any $\mathbf{x} \in S^{m-1}$.

It is evident that a regular QSO V is ergodic; however, regularity does not follow from the ergodicity.

The following notations will be used in this paper. Let the interior of S^{m-1} be the set $\mathrm{int}\, S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_1 x_2 \cdots x_m > 0\}$; let $\partial S^{m-1} = S^{m-1} \setminus \mathrm{int}\, S^{m-1}$ denote the boundary of S^{m-1}; a face of the simplex S^{m-1} be the set $\Gamma_\theta = \left\{ \mathbf{x} \in S^{m-1} : x_k = 0, \ k \notin \theta \subset E_m \right\}$; let $\mathbf{e}_i = (\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{m,i}) \in S^{m-1}, \ i \in E_m$, denote the vertices of the simplex S^{m-1}, where $\delta_{i,j}$ is the Kronecker delta.

Consider the following quadratic stochastic operator defined on the S^2

$$V : \begin{cases} x_1' = bx_1^2 + \beta x_2^2 + x_1 x_2 + x_1 x_3, \\ x_2' = \alpha x_2^2 + cx_3^2 + x_1 x_2 + x_2 x_3, \\ x_3' = ax_1^2 + dx_3^2 + x_1 x_3 + x_2 x_3, \end{cases} \tag{4}$$

where $0 \le a, b, \alpha, \beta, c, d \le 1$, and $a + b = c + d = \alpha + \beta = 1$.

Using $x_1 + x_2 + x_3 = 1$ we rewrite the QSO (4) in the form

$$V : \begin{cases} x_1' = x_1 + \beta x_2^2 - ax_1^2 \\ x_2' = x_2 + cx_3^2 - \beta x_2^2 \\ x_3' = x_3 + ax_1^2 - cx_3^2, \end{cases} \tag{5}$$

where $0 \le a, \beta, c \le 1$ and $a + b = \alpha + \beta = c + d = 1$.

Notice that if $a = \beta = c = 0$ then corresponding QSO (5) it the identity map and otherwise it is a non-Volterra QSO. So the case be all parameters are equal was studied in [19]. Hence in the below we consider the cases when at least one of the coefficients a, β, c is non-zero and all parameters don't equal.

3. Fixed points

A fixed point of (5) is a solution of the equation $V(\mathbf{x}) = \mathbf{x}$ or, in other words, of the system

$$\begin{cases} x_1 = x_1 + \beta x_2^2 - a x_1^2, \\ x_2 = x_2 + c x_3^2 - \beta x_2^2, \\ x_3 = x_3 + a x_1^2 - c x_3^2. \end{cases} \tag{6}$$

For $\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta} > 0$ we denote $\mathbf{x}^* := \mathbf{x}^*(a, \beta, c) = (x_1^*, x_2^*, x_3^*)$, where

$$x_1^* = \frac{\sqrt{c\beta}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}, \quad x_2^* = \frac{\sqrt{ac}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}, \quad x_3^* = \frac{\sqrt{a\beta}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}.$$

Theorem 3.1. *For the QSO* (5) *the following statements are hold:*

i)

$$\mathrm{Fix}\,(V) = \begin{cases} \Gamma_{\{2,3\}}, & a > 0, \ \beta = c = 0, \\ \Gamma_{\{1,3\}}, & \beta > 0, \ a = c = 0, \\ \Gamma_{\{1,2\}}, & c > 0, \ a = \beta = 0, \\ \{\mathbf{e}_1\}, & a = 0, \ c, \beta > 0, \\ \{\mathbf{e}_2\}, & \beta = 0, \ a, c > 0, \\ \{\mathbf{e}_3\}, & c = 0, \ a, \beta > 0, \\ \{\mathbf{x}^*\}, & a > 0, \beta > 0, c > 0; \end{cases} \tag{7}$$

ii) *If* $a > 0, \beta > 0, c > 0$ *then* $\mathbf{x}^*$ *is an attracting fixed point, in other cases any fixed point has the non-hyperbolic type.*

Proof. i) Let $a > 0, \beta = c = 0$ then the system (6) has following form

$$\begin{cases} x_1 = x_1 - a x_1^2 \\ x_2 = x_2 \\ x_3 = x_3 + a x_1^2. \end{cases} \tag{8}$$

It is easy to see that the first (third) equation of (8) has a unique solution $x_1 = 0$. Therefore it follows that any point of the face $\Gamma_{\{23\}}$ is a solution of the system (8).

The cases $a = c = 0, \ \beta \neq 0$ and $a = \beta = 0, \ c \neq 0$ can be considered similarly.

Let $a = 0, \ \beta > 0, c > 0$ then the system (6) has following form

$$\begin{cases} x_1 = x_1 + \beta x_2^2, \\ x_2 = x_2 + c x_3^2 - \beta x_2^2, \\ x_3 = x_3 - c x_3^2. \end{cases} \tag{9}$$

It is easy to see that the solution of the system (9) is $x_2 = x_3 = 0$, $x_1 = 1$, i.e. we get the vertex $\mathbf{e}_1$.

The cases $\beta = 0, \ a > 0, c > 0$, and $c = 0, \ a > 0, \beta > 0$ can be considered similarly.

Let $a\beta + ac + \beta c > 0$ then it follows that at least two of the coefficients a, β, c are non-zero. For the definiteness we suppose that $a > 0$ and $\beta > 0$. Then from the system (6) we have

$$a x_1^2 = \beta x_2^2 = c x_3^2 \quad \Rightarrow \quad x_1 = \sqrt{\frac{c}{a}} x_3, \quad x_2 = \sqrt{\frac{c}{\beta}} x_3.$$

Substituting them to $x_1 + x_2 + x_3 = 1$ one has

$$\sqrt{\frac{c}{a}} x_3 + \sqrt{\frac{c}{\beta}} x_3 + x_3 = 1 \quad \Rightarrow \quad x_3 = \frac{\sqrt{a\beta}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}.$$

Therefore it follows

$$x_1 = \frac{\sqrt{c\beta}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}} \quad \text{and} \quad x_2 = \frac{\sqrt{ac}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}.$$

Similarly the same results one has in the cases $a > 0, c > 0$ and $\beta > 0, c > 0$.

Hence in this case we have $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$ is a unique solution of the system (6).

ii) Now we are going to find the types of the fixed points of the QSO (5). To find the type of a fixed point of the QSO (5) we rewrite it as follows:

$$\begin{cases} x_1' = x_1 + \beta x_2^2 - ax_1^2 \\ x_2' = cx_1^2 + 2cx_1x_2 + (c - \beta)x_2^2 - 2cx_1 + (1 - 2c)x_2 + c, \end{cases} \tag{10}$$

here, $(x_1, x_2) \in \{(x, y) : x, y \geq 0, \ 0 \leq x + y \leq 1\}$, and x_1, x_2 are the first two coordinates of a point belonging to the simplex S^2.

The Jacobi matrix has the form

$$DV(\mathbf{x}) = \begin{pmatrix} 1 - 2ax_1 & 2\beta x_2 \\ 2cx_1 + 2cx_2 - 2c & 2cx_1 + 2(c - \beta)x_2 + 1 - 2c \end{pmatrix}. \tag{11}$$

One has that the eigenvalues of the Jacobi matrix (11) are

$$\lambda_{1,2} = 1 - ax_1 - \beta x_2 - cx_3 \pm \sqrt{D}, \quad \text{where}$$

$$D \equiv D(a, \beta, c) = a^2x_1^2 + \beta^2x_2^2 + c^2x_3^2 - 2a\beta x_1x_2 - 2acx_1x_3 - 2\beta cx_2x_3.$$

Therefore from the last in the case $a = 0$, $\beta c > 0$ for the vertex $\mathbf{e}_1$ we have the eigenvalues $\lambda_{1,2} = 1$. Similarly if $\beta = 0$, $ac > 0$ (resp. $c = 0$, $a\beta > 0$) then for the vertex $\mathbf{e}_2$ (resp. $\mathbf{e}_3$) it follows the eigenvalues $\lambda_{1,2} = 1$. Consequently we obtain that the vertex $\mathbf{e}_1$ (resp.$\mathbf{e}_2, \mathbf{e}_3$) is a non-hyperbolic fixed point when $a = 0$, $c\beta > 0$ (resp. $\beta = 0$, $ac > 0$, $c = 0$, $a\beta > 0$).

In the case $a = \beta = 0, c > 0$ any point of the face $\Gamma_{\{1,2\}}$ is a fixed point of the QSO (5). For these fixed points the Jacobi matrix (11) has the eigenvalues $\lambda_{1,2} = 1$. Therefore any fixed point from the face $\Gamma_{\{1,2\}}$ is a non-hyperbolic point.

Similarly in the case $a > 0, \beta = c = 0$ (resp. $\beta > 0, a = c = 0$) any point of the face $\Gamma_{\{2,3\}}$ (resp. $\Gamma_{\{1,3\}}$) is a fixed point of the QSO (5) and one can show that this fixed point is a non-hyperbolic point.

In the case $a > 0, \beta > 0, c > 0$ the point $\mathbf{x}^*$ is a unique fixed point of the QSO (5)in the int S^2. So at the fixed point $\mathbf{x}^*$ the Jacobi matrix (11) has the following eigenvalues

$$\lambda_{1,2} = 1 - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} \right)$$

$$\pm \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}$$

$$= 1 - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} \pm \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}} \right).$$

The last equation shows that if one of the parameters a, β, c is zero then $\mathbf{x}^*$ is a non-hyperbolic point.

Let $a > 0$, $\beta > 0$, $c > 0$. Denote $B(a, \beta, c) = a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}$. It is easy to verify that if $a = \beta = c > 0$ then we have $B(a, a, a) = -3a < 0$. Also if we choose $a = 1/4, \beta = c = 1/256$ then it follows that $B(a, \beta, c) = 3/16 > 0$. Thus, $B(a, \beta, c)$ may assume both negative and non-negative values.

a) Assume that $B(a, \beta, c) < 0$ then it holds that $0 < |\lambda_{1,2}| < 1$. Indeed, evidently that the eigenvalues $\lambda_{1,2}$ are complex numbers. Therefore we get

$$|\lambda_{1,2}|^2 = \left(1 - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)\right)^2$$
$$+ \left(\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right)^2 > 0.$$

$$|\lambda_{1,2}|^2 = 1 - \frac{2\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)$$
$$+ \frac{a\beta c}{\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)^2}\left(a + \beta + c + 2\sqrt{a\beta} + 2\sqrt{ac} + 2\sqrt{\beta c}\right)$$
$$+ \frac{a\beta c}{\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)^2}\left(a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}\right)$$
$$= 1 - \frac{2\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)$$
$$+ \frac{2a\beta c}{\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)^2}\left(a + \beta + c\right) \Rightarrow$$

$$|\lambda_{1,2}|^2 = 1 - \frac{2\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(a + \beta + c\right)\right) < 1,$$

where we have used the relation

$$\sqrt{a} + \sqrt{\beta} + \sqrt{c} - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(a + \beta + c\right) = \sqrt{a}\left(1 - \frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\right)$$
$$+ \sqrt{\beta}\left(1 - \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\right) + \sqrt{c}\left(1 - \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\right) > 0.$$

Because, for all $a, \beta, c \in (0, 1]$ it hold

$$0 < \frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1, \quad 0 < \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1, \quad 0 < \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1. \tag{12}$$

Indeed from inequalities $0 < a \leq 1$ and $0 < \frac{\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1$ we have $0 < \frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1$.

Similarly can be proved the inequalities $0 < \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1$ and $0 < \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < 1$.

Consequently we obtain $0 < |\lambda_{1,2}| < 1$ which imply that the unique fixed point $\mathbf{x}^*$ is an attracting point when $B(a, \beta, c) < 0$.

b) Suppose that $B(a, \beta, c) = 0$ then it holds $a + \beta + c = 2\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)$ and using it we have

$$\lambda_{1,2} = 1 - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)$$
$$\leq 1 - \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)$$
$$= 1 - \sqrt{a\beta c} < 1.$$

Using (12) it is easy to verify that

$$\frac{a\sqrt{\beta c} + \beta\sqrt{ac} + c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} - 1 = \frac{a\sqrt{\beta c} + \beta\sqrt{ac} + c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} - \frac{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}$$
$$= \frac{(a - 1)\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{(\beta - 1)\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{(c - 1)\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \leq 0$$

for all $a, \beta, c \in (0, 1]$. It is easy to check that the equality holds if and only if when $a = \beta = c = 1$. But in the case $a = \beta = c = 1$ one has that $B(a, \beta, c) = -3 < 0$ and it contradicts to our assumption $B(a, \beta, c) = 0$. Therefore the equality case impossible and we have that

$$\lambda_{1,2} = 1 - \frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} - \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} - \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} > 0.$$

Hence we obtain $0 < |\lambda_{1,2}| < 1$ and it follows that the fixed point $\mathbf{x}^*$ is an attracting point when $B(a, \beta, c) = 0$.

c) Suppose that $B(a, \beta, c) > 0$ then it holds $a + \beta + c > 2\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)$. It is easy to see that

$$\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} + \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right) > 0$$

for all $a, \beta, c \in (0, 1]$. Since $a + \beta + c > 2\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)$ it follows that

$$\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} + \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right) < 1,$$

where we have used

$$\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} + \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right)$$

$$< \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} + \sqrt{a + \beta + c + 2\sqrt{a\beta} + 2\sqrt{ac} + 2\sqrt{\beta c}}\right)$$

$$= \frac{2\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)$$

$$= 2\left(\frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}}\right) \leq 2.$$

for all $a, \beta, c \in (0, 1]$ and the last equality holds if and only if when $a = \beta = c = 1$. But in the case $a = \beta = c = 1$ one has that $B(a, \beta, c) = -3 < 0$ and it contradicts our assumption $B(a, \beta, c) > 0$. Therefore it follows that $-1 < \lambda_1 < 1$ and we have $0 < |\lambda_1| < 1$.

For all $a, \beta, c \in (0, 1]$ it holds

$$\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)^2 - \left(a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}\right) = 4\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right) > 0$$

and consequently it follows

$$\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} - \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right) > 0.$$

Since $a + \beta + c > 2\left(\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}\right)$ we get

$$\frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} - \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right) < 1,$$

where we have used

$$\frac{\sqrt{a\beta c}\left(\sqrt{a} + \sqrt{\beta} + \sqrt{c} - \sqrt{a + \beta + c - 2\sqrt{a\beta} - 2\sqrt{ac} - 2\sqrt{\beta c}}\right)}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} < \frac{\sqrt{a\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \left(\sqrt{a} + \sqrt{\beta} + \sqrt{c}\right)$$

$$= \frac{a\sqrt{\beta c}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{\beta\sqrt{ac}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} + \frac{c\sqrt{a\beta}}{\sqrt{a\beta} + \sqrt{ac} + \sqrt{\beta c}} \leq 1$$

for all $a, \beta, c \in (0, 1]$. Therefore it follows that $0 < \lambda_2 < 1$ and we have $0 < |\lambda_2| < 1$. Consequently we obtain $0 < |\lambda_{1,2}| < 1$ and it follows that the unique fixed point $\mathbf{x}^*$ is an attracting point when $B(a, \beta, c) > 0$.

Thus we have that if $a > 0, \beta > 0, c > 0$ then the unique fixed point $\mathbf{x}^*$ is an attracting point.

The proof of Theorem 3.1 is complete. □

4. The ω - limit set

The problem of describing the ω - limit set of a trajectory is of great importance in the theory of dynamical systems. In this section we solve this problem for the QSO (5).

Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in S^2$ be the initial point and let $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ be the trajectory of $\mathbf{x}^{(0)}$ under the action of the operator (5), that is,

$$\mathbf{x}^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right) = V\left(\mathbf{x}^{(n-1)}\right), \quad n = 1, 2, \dots$$

Theorem 4.1. *For the QSO* (5) *we have*

$$\omega_V\left(\mathbf{x}^{(0)}\right) = \begin{cases} \left\{\left(x_1^{(0)}, 1 - x_1^{(0)}, 0\right)\right\}, & if \ a = \beta = 0,\ c > 0,\ \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{12}, \\[2mm] \left\{\left(1 - x_3^{(0)}, 0, x_3^{(0)}\right)\right\}, & if \ a = c = 0,\ \beta > 0,\ \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{13}, \\[2mm] \left\{\left(0, x_2^{(0)}, 1 - x_2^{(0)}\right)\right\}, & if \ \beta = c = 0,\ a > 0,\ \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{23}, \\[2mm] \{\mathbf{e}_1\}, & if \ a = 0, \beta > 0, c > 0,\ \mathbf{x}^{(0)} \in S^2, \\[2mm] \{\mathbf{e}_2\}, & if \ \beta = 0, a > 0, c > 0,\ \mathbf{x}^{(0)} \in S^2, \\[2mm] \{\mathbf{e}_3\}, & if \ c = 0, a > 0, \beta > 0,\ \mathbf{x}^{(0)} \in S^2, \\[2mm] \{\mathbf{x}^*\}, & if \ a > 0, \beta > 0, c > 0,\ \mathbf{x}^{(0)} \in S^2. \end{cases}$$

Proof. i) Let $a = \beta = 0$, $c > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1, \\ x_2' = x_2 + c x_3^2, \\ x_3' = x_3 - c x_3^2. \end{cases} \tag{13}$$

Due to Theorem 3.1 any point of the face $\Gamma_{\{12\|}$ is a fixed point and this fixed point is a non-hyperbolic point. It is easy to see that

$$x_1' = x_1, \quad x_2' \geq x_2, \quad x_3' \leq x_3 \quad \text{for all} \quad \mathbf{x} \in S^2.$$

So for any initial point $\mathbf{x}^{(0)} \in S^2$ we have

$$x_1^{(n+1)} = x_1^{(0)}, \quad x_2^{(n+1)} \geq x_2^{(n)}, \quad x_3^{(n+1)} \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore it follows that the sequence $\left\{x_3^{(n)}\right\}_{n=0,1,2,\dots}$ (resp. $\left\{x_2^{(n)}\right\}_{n=0,1,2,\dots}$) is a decreasing (resp. an increasing) and bounded from the below (resp. above) sequence. Consequently there are the following limits

$$\lim_{n\to\infty} x_1^{(n)} = x_1^{(0)}, \quad \lim_{n\to\infty} x_2^{(n)} = \xi, \quad \lim_{n\to\infty} x_3^{(n)} = \eta.$$

We claim that $\eta = 0$. Suppose on the contrary, that is let $\eta > 0$. Then one has

$$1 = \lim_{n\to\infty} \frac{x_3^{(n+1)}}{x_3^{(n)}} = \lim_{n\to\infty} \left(1 - c x_3^{(n)}\right) = 1 - c\eta < 1.$$

The last relation is a contradiction. Therefore it follows that

$$\lim_{n\to\infty} x_1^{(n)} = x_1^{(0)}, \quad \lim_{n\to\infty} x_2^{(n)} = 1 - x_1^{(0)}, \quad \lim_{n\to\infty} x_3^{(n)} = 0.$$

Thus we obtain that for any initial point $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in S^2 \setminus \Gamma_{12}$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ converges to the fixed point $\left(x_1^{(0)}, 1 - x_1^{(0)}, 0\right)$, that is

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \left(x_1^{(0)}, 1 - x_1^{(0)}, 0\right).$$

ii) Let $a = c = 0$, $\beta > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1 + \beta x_2^2, \\ x_2' = x_2 - \beta x_2^2, \\ x_3' = x_3. \end{cases} \tag{14}$$

Due to Theorem 3.1 any point of the face Γ_{13} is a fixed point. Moreover any fixed point is a non-hyperbolic point.

Repeating the methods and techniques of the part i) one can prove that for any initial point $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \right) \in S^2 \setminus \Gamma_{13}$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ converges to the fixed point $\left(1 - x_3^{(0)}, 0, x_3^{(0)} \right)$, that is

$$\lim_{n \to \infty} \mathbf{x}^{(n)} = \left(1 - x_3^{(0)}, 0, x_3^{(0)} \right).$$

iii) Let $\beta = c = 0$, $a > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1 - a x_1^2, \\ x_2' = x_2, \\ x_3' = x_3 + a x_1^2. \end{cases} \tag{15}$$

Due to Theorem 3.1 any point from the face Γ_{23} is a fixed point. Moreover any fixed point is a non-hyperbolic point.

Repeating the methods and techniques of the part i) one can show that for any initial $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \right) \in S^2 \setminus \Gamma_{23}$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ converges to the fixed point $\left(0, x_2^{(0)}, 1 - x_2^{(0)} \right)$, that is

$$\lim_{n \to \infty} \mathbf{x}^{(n)} = \left(0, x_2^{(0)}, 1 - x_2^{(0)} \right).$$

iv) Let $a = 0, \beta > 0, c > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1 + \beta x_2^2, \\ x_2' = x_2 + c x_3^2 - \beta x_2^2, \\ x_3' = x_3 - c x_3^2. \end{cases} \tag{16}$$

Due to Theorem 3.1 it follows that the vertex $\mathbf{e}_1$ is a unique fixed point and it is a non-hyperbolic point.

It is easy to see that $x_1' \geq x_1$, $x_3' \leq x_3$ for all $\mathbf{x} \in S^2$. So for any initial point $\mathbf{x}^{(0)} \in S^2$ we have

$$x_1^{(n+1)} \geq x_1^{(n)}, \quad x_3^{(n+1)} \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore it follows that the sequence $\left\{ x_3^{(n)} \right\}$ (resp. $\left\{ x_1^{(n)} \right\}$) is a decreasing (resp. an increasing) and bounded from the below (resp. above) sequence. Consequently it follows the existence the following limits

$$\lim_{n \to \infty} x_1^{(n)} = \xi, \quad \lim_{n \to \infty} x_3^{(n)} = \eta \quad \Rightarrow \quad \lim_{n \to \infty} x_2^{(n)} = 1 - \lim_{n \to \infty} x_1^{(n)} - \lim_{n \to \infty} x_3^{(n)} = 1 - \xi - \eta.$$

Using the latter from (16) we have

$$\begin{cases} \xi = \xi + \beta(1 - \xi - \eta)^2, \\ 1 - \xi - \eta = 1 - \xi - \eta + c\eta^2 - \beta(1 - \xi - \eta)^2, \\ \eta = \eta - c\eta^2. \end{cases} \tag{17}$$

The third equation of (17) imply that $\eta = 0$ and using it from the first equation of (17) we have $\xi = 1$. Thus we obtain that for any initial point $\mathbf{x}^{(0)} \in S^2$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ converges to the vertex $\mathbf{e}_1$, that is

$$\lim_{n \to \infty} \mathbf{x}^{(n)} = \mathbf{e}_1 = (1, 0, 0).$$

v) Let $\beta = 0, a > 0, c > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1 - a x_1^2, \\ x_2' = x_2 + c x_3^2, \\ x_3' = x_3 + a x_1^2 - c x_3^2. \end{cases} \tag{18}$$

Due to Theorem 3.1 the vertex $\mathbf{e}_2$ is a unique fixed point and it is non-hyperbolic point.

Repeating the methods and techniques of the part iv) one can show that for any initial point $\mathbf{x}^{(0)} \in S^2$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\ldots}$ converges to the vertex $\mathbf{e}_2$, that is

$$\lim_{n \to \infty} \mathbf{x}^{(n)} = \mathbf{e}_2 = (0,1,0).$$

vi) Let $c = 0, a > 0, \beta > 0$. Then the QSO (5) has the following form

$$V : \begin{cases} x_1' = x_1 + \beta x_2^2 - ax_1^2, \\ x_2' = x_2 - \beta x_2^2, \\ x_3' = x_3 + ax_1^2. \end{cases} \tag{19}$$

Due to Theorem 3.1 the vertex $\mathbf{e}_3$ is a unique fixed point and it is a non-hyperbolic point.

Repeating the methods and techniques of the part iv) one can show that for any initial point $\mathbf{x}^{(0)} \in S^2$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\ldots}$ converges to vertex $\mathbf{e}_3$, that is

$$\lim_{n \to \infty} \mathbf{x}^{(n)} = \mathbf{e}_3 = (0,0,1).$$

vii) Let $a > 0, \beta > 0, c > 0$. Then the QSO (5) has the following form

$$W : \begin{cases} x_1' = x_1 + \beta x_2^2 - ax_1^2 \\ x_2' = x_2 + cx_3^2 - \beta x_2^2 \\ x_3' = x_3 + ax_1^2 - cx_3^2. \end{cases} \tag{20}$$

Due to Theorem 3.1 the point $\mathbf{x}^*$ is a unique fixed point. Besides, the fixed point $\mathbf{x}^*$ is an attracting point. It is easy to see that

$$\mathbf{x}^* = \left(\frac{C}{\sqrt{a}}, \frac{C}{\sqrt{\beta}}, \frac{C}{\sqrt{c}} \right), \quad \text{where} \quad C = \frac{\sqrt{ac\beta}}{\sqrt{\beta c} + \sqrt{ac} + \sqrt{a\beta}}.$$

Consider the following function

$$\varphi(\mathbf{x}) = \max \left\{ \sqrt{a} \, |x_1 - x_1^*|, \sqrt{\beta} \, |x_2 - x_2^*|, \sqrt{c} \, |x_3 - x_3^*| \right\}.$$

Let $\mathbf{x}^{(0)} \in S^2$. Then for n-th step of the trajectory we have

$$\varphi\left(\mathbf{x}^{(n)}\right) = \max \left\{ \sqrt{a} \left| x_1^{(n)} - x_1^* \right|, \sqrt{\beta} \left| x_2^{(n)} - x_2^* \right|, \sqrt{c} \left| x_3^{(n)} - x_3^* \right| \right\}, \quad n = 0,1,2,$$

Consider the following differences

$$\begin{aligned} x_1^{(n+1)} - x_1^* &= x_1^{(n)} - x_1^* + \left(\sqrt{\beta} x_2^{(n)} - \sqrt{a} x_1^{(n)} \right)\left(\sqrt{\beta} x_2^{(n)} + \sqrt{a} x_1^{(n)} \right) \\ &= x_1^{(n)} - x_1^* + \left(\left(\sqrt{\beta} x_2^{(n)} - C \right) - \left(\sqrt{a} x_1^{(n)} - C \right) \right)\left(\sqrt{\beta} x_2^{(n)} + \sqrt{a} x_1^{(n)} \right) \\ &= \left(x_1^{(n)} - x_1^* \right)\left(1 - \sqrt{a} \left(\sqrt{\beta} x_2^{(n)} + \sqrt{a} x_1^{(n)} \right) \right) + \left(x_2^{(n)} - x_2^* \right)\sqrt{\beta} \left(\sqrt{\beta} x_2^{(n)} + \sqrt{a} x_1^{(n)} \right), \end{aligned} \tag{21}$$

$$\begin{aligned} x_2^{(n+1)} - x_2^* &= x_2^{(n)} - x_2^* + \left(\sqrt{c} x_3^{(n)} - \sqrt{\beta} x_2^{(n)} \right)\left(\sqrt{c} x_3^{(n)} + \sqrt{\beta} x_2^{(n)} \right) \\ &= x_2^{(n)} - x_2^* + \left(\left(\sqrt{c} x_3^{(n)} - C \right) - \left(\sqrt{\beta} x_2^{(n)} - C \right) \right)\left(\sqrt{c} x_3^{(n)} + \sqrt{\beta} x_2^{(n)} \right) \\ &= \left(x_2^{(n)} - x_2^* \right)\left(1 - \sqrt{\beta} \left(\sqrt{c} x_3^{(n)} + \sqrt{\beta} x_2^{(n)} \right) \right) + \left(x_3^{(n)} - x_3^* \right)\sqrt{c} \left(\sqrt{c} x_3^{(n)} + \sqrt{\beta} x_2^{(n)} \right), \end{aligned} \tag{22}$$

$$\begin{aligned} x_3^{(n+1)} - x_3^* &= x_3^{(n)} - x_3^* + \left(\sqrt{a} x_1^{(n)} - \sqrt{c} x_3^{(n)} \right)\left(\sqrt{a} x_1^{(n)} + \sqrt{c} x_3^{(n)} \right) \\ &= x_3^{(n)} - x_3^* + \left(\left(\sqrt{a} x_1^{(n)} - C \right) - \left(\sqrt{c} x_3^{(n)} - C \right) \right)\left(\sqrt{a} x_1^{(n)} + \sqrt{c} x_3^{(n)} \right) \\ &= \left(x_3^{(n)} - x_3^* \right)\left(1 - \sqrt{c} \left(\sqrt{a} x_1^{(n)} + \sqrt{c} x_3^{(n)} \right) \right) + \left(x_1^{(n)} - x_1^* \right)\sqrt{a} \left(\sqrt{a} x_1^{(n)} + \sqrt{c} x_3^{(n)} \right). \end{aligned} \tag{23}$$

From the differences (21),(22) and (23) we have

$$\sqrt{a}\left|x_1^{(n+1)} - x_1^*\right| \leq \max\left\{\sqrt{a}\left|x_1^{(n)} - x_1^*\right|, \sqrt{\beta}\left|x_2^{(n)} - x_2^*\right|\right\},$$

$$\sqrt{\beta}\left|x_2^{(n+1)} - x_2^*\right| \leq \max\left\{\sqrt{\beta}\left|x_2^{(n)} - x_2^*\right|, \sqrt{c}\left|x_3^{(n)} - x_3^*\right|\right\},$$

$$\sqrt{c}\left|x_3^{(n+1)} - x_3^*\right| \leq \max\left\{\sqrt{c}\left|x_3^{(n)} - x_3^*\right|, \sqrt{a}\left|x_1^{(n)} - x_1^*\right|\right\}.$$

Let us show the first one

$$\begin{aligned}
\sqrt{a}|x_1^{(n+1)} - x_1^*| &= \sqrt{a}\left|x_1^{(n)} - x_1^* + \left(\sqrt{\beta}x_2^{(n)} - \sqrt{a}x_1^{(n)}\right)\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right| \\
&= \sqrt{a}\left|x_1^{(n)} - x_1^* + \left(\left(\sqrt{\beta}x_2^{(n)} - C\right) - \left(\sqrt{a}x_1^{(n)} - C\right)\right)\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right| \\
&= \sqrt{a}\left|\left(x_1^{(n)} - x_1^*\right)\left(1 - \sqrt{a}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right) + \left(x_2^{(n)} - x_2^*\right)\sqrt{\beta}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right| \\
&\leq \sqrt{a}\left|x_1^{(n)} - x_1^*\right|\left(1 - \sqrt{a}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right) + \sqrt{\beta}\left|x_2^{(n)} - x_2^*\right|\sqrt{a}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right) \\
&\leq \theta\left(1 - \sqrt{a}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right) + \sqrt{a}\left(\sqrt{\beta}x_2^{(n)} + \sqrt{a}x_1^{(n)}\right)\right) = \theta,
\end{aligned} \tag{24}$$

where $\theta = \max\left\{\sqrt{a}\left|x_1^{(n)} - x_1^*\right|, \sqrt{\beta}\left|x_2^{(n)} - x_2^*\right|\right\}$.

The rest cases can be considered in a similar manner.

Therefore it follows $\varphi\left(\mathbf{x}^{(n+1)}\right) \leq \varphi\left(\mathbf{x}^{(n)}\right)$, i.e. the function $\varphi(\mathbf{x})$ is a Lyapunov function for the QSO (20).

For the convenience in further calculation we denote $a_1 = a$, $a_2 = \beta$, $a_3 = c$. Then we have

$$\varphi(\mathbf{x}) = \max_{j \in \{1,2,3\}}\left\{\sqrt{a_j}\left|x_j - x_j^*\right|\right\} \Rightarrow \varphi(\mathbf{x}) = \max_{j \in \{1,2,3\}}\left\{\sqrt{a_j}\left|x_j - \frac{C}{\sqrt{a_j}}\right|\right\}.$$

It is clear that $\varphi(\mathbf{x}) = 0$ iff $\mathbf{x} = \mathbf{x}^*$ in the simplex S^2.

Denote $A_{ij}^{(n)} = \sqrt{a_i}\left(\sqrt{a_j}x_j^{(n)} + \sqrt{a_i}x_i^{(n)}\right)$, $i, j = 1, 2, 3$. It is clear that for any $n \in \mathbb{N}$ and $i, j = 1, 2, 3$ it holds

$$0 \leq A_{ij}^{(n)} \leq 1.$$

If $a_1a_2 + a_1a_3 + a_2a_3 > 0$ and $a_1 \neq 1, a_2 \neq 1, a_3 \neq 1$, then $0 < A_{ij}^{(n)} < 1$. The case $a_1 = a_2 = a_3 = 1$ is studied in [19].

For any $0 < a_1, a_2, a_3 < 1$ and for any initial $\mathbf{x}^{(0)} \in \text{int } S^2$ from (20) we have

$$\varphi\left(\mathbf{x}^{(n+1)}\right) \leq \varphi\left(\mathbf{x}^{(n)}\right) \leq \cdots \leq \varphi\left(\mathbf{x}^{(0)}\right).$$

Therefore we have that the sequence $\left\{\varphi\left(\mathbf{x}^{(n)}\right)\right\}_{n \geq 0}$ is a decreasing and bounded from the below sequence. So it follows that there exists the limit $\lim_{n \to \infty} \varphi\left(\mathbf{x}^{(n)}\right) = \tau \geq 0$. We claim that $\tau = 0$. Let us suppose on the contrary, that is $\tau > 0$. Then we have that

$$\begin{aligned}
1 &= \lim_{n \to \infty}\frac{\varphi(\mathbf{x}^{(n+1)})}{\varphi(\mathbf{x}^{(n)})} = \lim_{n \to \infty}\left|\frac{\sqrt{a_{j_m}}\left(x_{j_m}^{(n)} - \frac{C}{\sqrt{a_{j_m}}}\right)\left(1 - A_{j_m j_l}^{(n)}\right) + \sqrt{a_{j_l}}\left(x_{j_l}^{(n)} - \frac{C}{\sqrt{a_{j_l}}}\right)A_{j_m j_l}^{(n)}}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n)} - \frac{C}{\sqrt{a_{j_k}}}\right)}\right| \\
&= \lim_{n \to \infty}\left|\frac{\sqrt{a_{j_m}}\left(x_{j_m}^{(n)} - \frac{C}{\sqrt{a_{j_m}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n)} - \frac{C}{\sqrt{a_{j_k}}}\right)}\left(1 - A_{j_m j_l}^{(n)}\right) + \frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n)} - \frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n)} - \frac{C}{\sqrt{a_{j_k}}}\right)}A_{j_m j_l}^{(n)}\right|.
\end{aligned} \tag{25}$$

We consider all possible cases:

I. Suppose that for a subsequence $\{n_t\}_{t=0,1,2,\ldots} \subset \mathbb{N}$ it hold

$$\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right| < \sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|, \quad \sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right| < \sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|.$$

Then one has (25),

$$
\begin{aligned}
1 &= \lim_{t\to\infty}\left|\frac{\sqrt{a_{j_m}}\left(x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right)}\left(1 - A_{j_m j_l}^{(n_t)}\right) + \frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right)}A_{j_m j_l}^{(n_t)}\right| \\
&\leq \lim_{t\to\infty}\left(\frac{\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|}\left(1 - A_{j_m j_l}^{(n_t)}\right) + \frac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|}A_{j_m j_l}^{(n_t)}\right) \\
&\leq \lim_{t\to\infty}\frac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|},
\end{aligned}
$$

where we have used the following our assumption

$$\frac{\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|} < \frac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|}. \tag{26}$$

Since $\dfrac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|} < 1$ it follows that $\displaystyle\lim_{t\to\infty}\frac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|} = 1. \tag{27}$

Using the latter from (25), (26) and (27) we have

$$1 \leq \lim_{t\to\infty}\left(\frac{\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|}\left(1 - A_{j_m j_l}^{(n_t)}\right) + \frac{\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|}A_{j_m j_l}^{(n_t)}\right) \Rightarrow$$

$$\lim_{t\to\infty}\frac{\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right|}{\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|} = 1.$$

Consequently it follows that

$$\lim_{t\to\infty}\sqrt{a_{j_k}}\left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right| = \lim_{t\to\infty}\sqrt{a_{j_m}}\left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right| = \lim_{t\to\infty}\sqrt{a_{j_l}}\left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right| = \tau.$$

The last equation holds in the one of the two the following cases:

a) Assume from $x_{j_k}^{(n_t)} \to \dfrac{C}{\sqrt{a_{j_k}}} \pm \dfrac{\tau}{\sqrt{a_{j_k}}}$ as $t \to \infty$ one has that

$$x_{j_m}^{(n_t)} \to \frac{C}{\sqrt{a_{j_m}}} \mp \frac{\tau}{\sqrt{a_{j_m}}} \quad \text{and} \quad x_{j_l}^{(n_t)} \to \frac{C}{\sqrt{a_{j_l}}} \mp \frac{\tau}{\sqrt{a_{j_l}}} \quad \text{as} \ t \to \infty.$$

Since $x_{j_k}^{(n_t)} + x_{j_m}^{(n_t)} + x_{j_l}^{(n_t)} = \dfrac{C}{\sqrt{a_{j_k}}} + \dfrac{C}{\sqrt{a_{j_m}}} + \dfrac{C}{\sqrt{a_{j_l}}} = 1$ as $t \to \infty$ it follows that

$$\sqrt{a_{j_m} a_{j_l}} = \sqrt{a_{j_k} a_{j_m}} + \sqrt{a_{j_k} a_{j_l}}.$$

Using the last equation and notation of x^* one has that it should be

$$\frac{C}{\sqrt{a_{j_k}}} = \frac{1}{2} \quad \Rightarrow \quad \frac{C}{\sqrt{a_{j_m}}} + \frac{C}{\sqrt{a_{j_l}}} = \frac{1}{2}.$$

It is easy to see that j_k in any step of the trajectory may take one element from the set $\{1, 2, 3\}$. Therefore the sequence $\left\{x_{j_k}^{(n_t)}\right\}$ of the trajectory corresponding taken values of j_k converges to $\dfrac{1}{2} \pm \dfrac{\tau}{\sqrt{a_{j_k}}}$. Another side since the values $\dfrac{C}{\sqrt{a_{j_k}}}, \dfrac{C}{\sqrt{a_{j_m}}}, \dfrac{C}{\sqrt{a_{j_l}}}$ are the coordinates fixed point it follows that the sequence $\left\{x_{j_k}^{(n_t)}\right\}$ of the trajectory corresponding taken values of j_k must be iteration only one coordinate. Using the limit

$$\lim_{t \to \infty} \frac{\sqrt{a_{j_l}} \left(x_{j_l}^{(n_t)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}} \left(x_{j_k}^{(n_t)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)} = -1$$

from $(20),(21),(22)$ we have

$$1 = \lim_{t \to \infty} \left| \frac{\sqrt{a_{j_k}} \left(x_{j_k}^{(n_t)} - \dfrac{1}{2}\right)\left(1 - A_{j_k j_l}^{(n_t)}\right) + \sqrt{a_{j_l}} \left(x_{j_l}^{(n_t)} - \dfrac{C}{\sqrt{a_{j_l}}}\right) A_{j_k j_l}^{(n_t)}}{\sqrt{a_{j_k}} \left(x_{j_k}^{(n_t)} - \dfrac{1}{2}\right)} \right|$$

and it contradicts to the our assumption

$$\sqrt{a_{j_m}} \left|x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right| < \sqrt{a_{j_k}} \left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|, \quad \sqrt{a_{j_l}} \left|x_{j_l}^{(n_t)} - \frac{C}{\sqrt{a_{j_l}}}\right| < \sqrt{a_{j_k}} \left|x_{j_k}^{(n_t)} - \frac{C}{\sqrt{a_{j_k}}}\right|.$$

b) Assume that from $x_{j_k}^{(n_t)} \to \dfrac{C}{\sqrt{a_{j_k}}} \pm \dfrac{\tau}{\sqrt{a_{j_k}}}$, as $n_t \to \infty$ one has that

$$x_{j_m}^{(n_t)} \to \frac{C}{\sqrt{a_{j_m}}} \pm \frac{\tau}{\sqrt{a_{j_m}}} \quad \text{and} \quad x_{j_l}^{(n_t)} \to \frac{C}{\sqrt{a_{j_l}}} \mp \frac{\tau}{\sqrt{a_{j_l}}}, \quad \text{as } t \to \infty.$$

In this case using the relation $x_{j_k}^{(n_t)} + x_{j_m}^{(n_t)} + x_{j_l}^{(n_t)} = \dfrac{C}{\sqrt{a_{j_k}}} + \dfrac{C}{\sqrt{a_{j_m}}} + \dfrac{C}{\sqrt{a_{j_l}}} = 1$ it follows that

$$\sqrt{a_{j_m} a_{j_k}} = \sqrt{a_{j_l} a_{j_m}} + \sqrt{a_{j_k} a_{j_l}}.$$

Again using form of x^* one has $\dfrac{C}{\sqrt{a_{j_l}}} = \dfrac{1}{2}$ and $\dfrac{C}{\sqrt{a_{j_m}}} + \dfrac{C}{\sqrt{a_{j_k}}} = \dfrac{1}{2}.$

As before j_l in any step of the trajectory may take one of the values $\{1, 2, 3\}$. Therefore the subsequence $\left\{x_{j_l}^{(n_t)}\right\}$ of the trajectory corresponding taken values of j_l converges to $\dfrac{1}{2} \pm \dfrac{\tau}{\sqrt{a_{j_l}}}$. Another side since the values $\dfrac{C}{\sqrt{a_{j_k}}}, \dfrac{C}{\sqrt{a_{j_m}}}, \dfrac{C}{\sqrt{a_{j_l}}}$ are the coordinates fixed point it follows that the sequence $\left\{x_{j_l}^{(n_t)}\right\}$ of the trajectory corresponding taken values of j_l must be iteration only one coordinate. But this coordinate cannot be maximum of the coordinates any step of the trajectory. Consequently from these facts and $(20),(21),(22)$ taking account that

$$\sqrt{a_{j_m}} \left(x_{j_m}^{(n_t)} - \frac{C}{\sqrt{a_{j_m}}}\right)(1 - A_{j_m j_l}^{(n_t)}) + \sqrt{a_{j_l}} \left(x_{j_l}^{(n_t)} - \frac{1}{2}\right) A_{j_m j_l}^{(n_t)} = \sqrt{a_{j_m}} \left(x_{j_m}^{(n_{t+1})} - \frac{C}{\sqrt{a_{j_m}}}\right)$$

it follows that $\sqrt{a_{j_m}}\left(x_{j_m}^{(n_{t+1})} - \dfrac{C}{\sqrt{a_{j_m}}}\right)$ is the maximum of the following values

$$\sqrt{a_{j_k}}\left(x_{j_k}^{(n_{t+1})} - \frac{C}{\sqrt{a_{j_k}}}\right), \quad \sqrt{a_{j_m}}\left(x_{j_m}^{((n+1)_t)} - \frac{C}{\sqrt{a_{j_m}}}\right), \quad \sqrt{a_{j_l}}\left(x_{j_l}^{(n_{t+1})} - \frac{C}{\sqrt{a_{j_l}}}\right).$$

But this contradicts to our assumption.

The case $x_{j_k}^{(n_t)} \to \dfrac{C}{\sqrt{a_{j_k}}} \pm \dfrac{\tau}{\sqrt{a_{j_k}}}, x_{j_l}^{(n_t)} \to \dfrac{C}{\sqrt{a_{j_l}}} \pm \dfrac{\tau}{\sqrt{a_{j_l}}}, x_{j_m}^{(n_t)} \to \dfrac{C}{\sqrt{a_{j_m}}} \pm \dfrac{\tau}{\sqrt{a_{j_m}}}$ impossible. Because in this case one has either

$$x_{j_k}^{(n_t)} + x_{j_l}^{(n_t)} + x_{j_m}^{(n_t)} \to 1 + \frac{\tau}{\sqrt{a_{j_k}}} + \frac{\tau}{\sqrt{a_{j_l}}} + \frac{\tau}{\sqrt{a_{j_m}}} > 1$$

or

$$x_{j_k}^{(n_t)} + x_{j_l}^{(n_t)} + x_{j_m}^{(n_t)} \to 1 - \frac{\tau}{\sqrt{a_{j_k}}} - \frac{\tau}{\sqrt{a_{j_l}}} - \frac{\tau}{\sqrt{a_{j_m}}} < 1.$$

II. Suppose that $j_m = j_k$ and for a subsequence $\{n_r\} \subset \mathbb{N}$ it holds $\sqrt{a_{j_l}}\left|x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right| < \sqrt{a_{j_k}}\left|x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right|$.
Then from (25) we have that

$$1 = \lim_{r\to\infty}\left|1 - A_{j_k j_l}^{(n_r)} + \frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)}A_{j_k j_l}^{(n)}\right|$$

$$= \lim_{r\to\infty}\left|1 + A_{j_k j_l}^{(n_r)}\left(\frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)} - 1\right)\right| < 1$$

If $\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)} \le 0$ then it follows that $\lim_{r\to\infty}\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)} = 1.$ \qquad (28)

If $\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_r)} - \dfrac{C}{\sqrt{a_{j_k}}}\right)} > 0$ then one has that $x_{j_k}^{(n_r)} \to \dfrac{C}{\sqrt{a_{j_k}}} \pm \dfrac{\tau}{\sqrt{a_{j_k}}}, \quad x_{j_l}^{(n_r)} \to \dfrac{C}{\sqrt{a_{j_l}}} \pm \dfrac{\tau}{\sqrt{a_{j_l}}}$ as $r \to \infty$

and the third coordinate converges to a $\tau' > 0$, that is, $x_{j_m}^{(n_r)} \to \dfrac{C}{\sqrt{a_{j_m}}} \mp \dfrac{\tau'}{\sqrt{a_{j_m}}}$ as $r \to \infty$. Therefore in this case the sequence $\left\{x_{j_k}^{(n_r)}\right\}$ of the trajectory corresponding taken values of j_k must be iteration only one coordinate. But this coordinate cannot be maximum of the coordinates any step of the trajectory and $\left\{x_{j_k}^{(n_r)}\right\}$ is the maximum in any step, that is

$$x_{j_k}^{((n+1)_r)} = x_{j_k}^{(n_r)} + a_{j_l}\left(x_{j_l}^{(n_r)}\right)^2 - a_{j_k}\left(x_{j_k}^{(n_r)}\right)^2.$$

And from the last equation it follows $\lim\limits_{r\to\infty}\sqrt{a_{j_k}}x_{j_k}^{(n_r)} = \lim\limits_{r\to\infty}\sqrt{a_{j_l}}x_{j_l}^{(n_r)}$ and $\lim\limits_{r\to\infty}x_{j_k}^{(n_{r+1})} = \lim\limits_{r\to\infty}x_{j_k}^{(n_r)}$. Also using the last results and (28) we have $\lim\limits_{r\to\infty}x_{j_l}^{(n_{r+1})} = \lim\limits_{r\to\infty}x_{j_l}^{(n_r)}$. So using the last and the relation $x_{j_l}^{(n_{r+1})} = x_{j_l}^{(n_r)} + a_{j_m}\left(x_{j_m}^{(n_r)}\right)^2 - a_{j_l}\left(x_{j_l}^{(n_r)}\right)^2$ one has that $\lim\limits_{r\to\infty}\sqrt{a_{j_l}}x_{j_l}^{(n_r)} = \lim\limits_{r\to\infty}\sqrt{a_{j_m}}x_{j_m}^{(n_r)}$, that is

$$\lim_{r\to\infty}\sqrt{a_{j_l}}x_{j_l}^{(n_r)} = \lim_{r\to\infty}\sqrt{a_{j_m}}x_{j_m}^{(n_r)} = \lim_{r\to\infty}\sqrt{a_{j_m}}x_{j_m}^{(n_r)} = C \pm \tau.$$

The last equation contradicts to the relation $\lim_{r\to\infty}\left(x_{j_k}^{(n_r)}+x_{j_m}^{(n_r)}+x_{j_l}^{(n_r)}\right)=1$.

The case $j_l=j_k$ and $\sqrt{a_{j_m}}\left|x_{j_m}^{(n_r)}-\dfrac{C}{\sqrt{a_{j_m}}}\right|<\sqrt{a_{j_k}}\left|x_{j_k}^{(n_r)}-\dfrac{C}{\sqrt{a_{j_k}}}\right|$ can be considered in a similar manner.

III. Let $j_m\neq j_k$ and for a subsequence $\{n_s\}_{s=0,1,2,\ldots}\subset\mathbb{N}$ it hold $\sqrt{a_{j_m}}\left|x_{j_m}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_m}}}\right|=\sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_k}}}\right|$,

$\sqrt{a_{j_l}}\left|x_{j_l}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_l}}}\right|<\sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_k}}}\right|$. If it holds $\dfrac{\sqrt{a_{j_m}}\left(x_{j_m}^{(n_s)}-\frac{C}{\sqrt{a_{j_m}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}=1$ then from (25) we obtain

$$1=\lim_{s\to\infty}\left|1-A_{j_mj_l}^{(n_s)}+\frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}A_{j_mj_l}^{(n_s)}\right|=\lim_{s\to\infty}\left|1+\left(\frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}-1\right)A_{j_mj_l}^{(n_s)}\right|<1,$$

because it is true $\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}<0$. If it holds $\dfrac{\sqrt{a_{j_m}}\left(x_{j_m}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}=-1$ then from (25) we get

$$1=\lim_{s\to\infty}\left|A_{j_mj_l}^{(n_s)}-1+\frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}A_{j_mj_l}^{(n_s)}\right|=\lim_{s\to\infty}\left|-1+\left(\frac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}+1\right)A_{j_mj_l}^{(n_s)}\right|<1.$$

If it holds $\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}>0$ then we have $\lim_{s\to\infty}\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}=-1$.

If it holds the inequality $\dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)}-\frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_s)}-\frac{C}{\sqrt{a_{j_k}}}\right)}<0$ that is, if it hold the following relations

$$x_{j_k}^{(n_s)}\to\frac{C}{\sqrt{a_{j_k}}}\pm\frac{\tau}{\sqrt{a_{j_k}}},\quad x_{j_m}^{(n_s)}\to\frac{C}{\sqrt{a_{j_m}}}\mp\frac{\tau}{\sqrt{a_{j_m}}},\quad x_{j_l}^{(n_s)}\to\frac{C}{\sqrt{a_{j_l}}}\mp\frac{\tau}{\sqrt{a_{j_l}}}\quad\text{as}\quad s\to\infty$$

then using $\lim_{s\to\infty}\left(x_{j_k}^{(n_s)}+x_{j_m}^{(n_s)}+x_{j_l}^{(n_s)}\right)=1$ we have $\sqrt{a_{j_m}a_{j_l}}=\sqrt{a_{j_l}a_{j_k}}+\sqrt{a_{j_k}a_{j_m}}$ and using the last it follows that

$$\frac{C}{\sqrt{a_{j_k}}}=\frac{1}{2}\quad\text{and}\quad\frac{C}{\sqrt{a_{j_m}}}+\frac{C}{\sqrt{a_{j_l}}}=\frac{1}{2}.$$

It is easy to see that j_k in any step of the trajectory may take one element from the set $\{1,2,3\}$. Therefore the subsequence $\left\{x_{j_k}^{(n_s)}\right\}$ of the trajectory corresponding taken values of j_k converges to $1/2$. Another side since the values $\dfrac{C}{\sqrt{a_{j_k}}},\dfrac{C}{\sqrt{a_{j_m}}},\dfrac{C}{\sqrt{a_{j_l}}}$ are the coordinates fixed point it follows that the subsequence $\left\{x_{j_k}^{(n_s)}\right\}$ of the trajectory corresponding taken values of j_k must be iteration only one coordinate. This is a contradiction to our assumptions $j_m\neq j_k$, $\sqrt{a_{j_m}}\left|x_{j_m}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_m}}}\right|=\sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)}-\dfrac{C}{\sqrt{a_{j_k}}}\right|$ and

$$\sqrt{a_{j_l}}\left|x_{j_l}^{(n_s)} - \frac{C}{\sqrt{a_{j_l}}}\right| < \sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)} - \frac{C}{\sqrt{a_{j_k}}}\right|, \text{ that is it does not hold the following equation}$$

$$\sqrt{a_{j_k}}\left(x_{j_k}^{(n_{s+1})} - \frac{1}{2}\right) = \sqrt{a_{j_m}}\left(x_{j_m}^{(n_s)} - \frac{C}{\sqrt{a_{j_m}}}\right)(1 - A_{j_m j_l}^{(n_s)}) + \sqrt{a_{j_l}}\left(x_{j_l}^{(n_s)} - \frac{C}{\sqrt{a_{j_l}}}\right)A_{j_m j_l}^{(n_s)}.$$

Similar results can be proved for the case $j_l \neq j_k$, $\sqrt{a_{j_l}}\left|x_{j_l}^{(n_s)} - \frac{C}{\sqrt{a_{j_l}}}\right| = \sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)} - \frac{C}{\sqrt{a_{j_k}}}\right|$ and

$$\sqrt{a_{j_m}}\left|x_{j_m}^{(n_s)} - \frac{C}{\sqrt{a_{j_m}}}\right| < \sqrt{a_{j_k}}\left|x_{j_k}^{(n_s)} - \frac{C}{\sqrt{a_{j_k}}}\right|.$$

IV. Suppose that for some subsequence $\{n_p\}_{p=0,1,2,\ldots} \subset \mathbb{N}$ it hold $\sqrt{a_{j_l}}\left|x_{j_l}^{(n_p)} - \frac{C}{\sqrt{a_{j_l}}}\right| = \sqrt{a_{j_k}}\left|x_{j_k}^{(n_p)} - \frac{C}{\sqrt{a_{j_k}}}\right|$

and $\sqrt{a_{j_m}}\left|x_{j_m}^{(n_p)} - \frac{C}{\sqrt{a_{j_m}}}\right| = \sqrt{a_{j_k}}\left|x_{j_k}^{(n_p)} - \frac{C}{\sqrt{a_{j_k}}}\right|$ and the coordinates j_k, j_m, j_l are different. Then it follows

$$\lim_{p\to\infty}\sqrt{a_{j_k}}\left|x_{j_k}^{(n_p)} - \frac{C}{\sqrt{a_{j_k}}}\right| = \lim_{p\to\infty}\sqrt{a_{j_l}}\left|x_{j_l}^{(n_p)} - \frac{C}{\sqrt{a_{j_l}}}\right| = \lim_{p\to\infty}\sqrt{a_{j_m}}\left|x_{j_m}^{(n_p)} - \frac{C}{\sqrt{a_{j_m}}}\right| = \tau$$

Suppose that if it holds $x_{j_k}^{(n_p)} \to \frac{C}{\sqrt{a_{j_k}}} \pm \frac{\tau}{\sqrt{a_{j_k}}}$ as $p \to \infty$ then one has

$$x_{j_m}^{(n_p)} \to \frac{C}{\sqrt{a_{j_m}}} \mp \frac{\tau}{\sqrt{a_{j_m}}} \quad \text{and} \quad x_{j_l}^{(n_p)} \to \frac{C}{\sqrt{a_{j_l}}} \mp \frac{\tau}{\sqrt{a_{j_l}}}, \quad \text{as} \quad p \to \infty.$$

In this case as proved in the a) of part I it does not hold the following equality

$$\sqrt{a_{j_k}}\left(x_{j_{k+1}}^{(n+1)_p} - \frac{1}{2}\right) = \sqrt{a_{j_m}}\left(x_{j_m}^{(n_p)} - \frac{C}{\sqrt{a_{j_m}}}\right)(1 - A_{j_m j_l}^{(n_p)}) + \sqrt{a_{j_l}}\left(x_{j_l}^{(n_p)} - \frac{C}{\sqrt{a_{j_l}}}\right)A_{j_m j_l}^{(n_p)}.$$

Analogously in the case when from $x_{j_k}^{(n_p)} \to \frac{C}{\sqrt{a_{j_k}}} \pm \frac{\tau}{\sqrt{a_{j_k}}}$ as $p \to \infty$ it follows that

$$x_{j_m}^{(n_p)} \to \frac{C}{\sqrt{a_{j_m}}} \pm \frac{\tau}{\sqrt{a_{j_m}}} \quad \text{and} \quad x_{j_l}^{(n_p)} \to \frac{C}{\sqrt{a_{j_l}}} \mp \frac{\tau}{\sqrt{a_{j_l}}} \quad \text{as} \quad p \to \infty$$

one has a contradiction to the assumption. In the all the rest cases also we have a contradiction to the relation

$$\lim_{p\to\infty}\left(x_{j_k}^{(n_p)} + x_{j_l}^{(n_p)} + x_{j_m}^{(n_p)}\right) = 1.$$

V. Suppose that $j_k = j_m$ and for some subsequnnce $n_q \in \mathbb{N}$ it hold

$$\sqrt{a_{j_l}}\left|x_{j_l}^{(n_q)} - \frac{C}{\sqrt{a_{j_l}}}\right| = \sqrt{a_{j_k}}\left|x_{j_k}^{(n_q)} - \frac{C}{\sqrt{a_{j_k}}}\right|, \sqrt{a_{j_m}}\left|x_{j_m}^{(n_q)} - \frac{C}{\sqrt{a_{j_m}}}\right| = \sqrt{a_{j_k}}\left|x_{j_k}^{(n_q)} - \frac{C}{\sqrt{a_{j_k}}}\right|.$$

If in this case it holds $\lim_{q\to\infty} \dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_q)} - \frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_q)} - \frac{C}{\sqrt{a_{j_k}}}\right)} = -1$ from (25) one has the following contradiction

$$1 = \lim_{q\to\infty}\left|1 - A_{j_k j_l}^{(n_q)} - A_{j_k j_l}^{(n_q)}\right| = \lim_{q\to\infty}\left|1 - 2A_{j_k j_l}^{(n_q)}\right| < 1.$$

If it holds that $\lim_{q\to\infty} \dfrac{\sqrt{a_{j_l}}\left(x_{j_l}^{(n_q)} - \frac{C}{\sqrt{a_{j_l}}}\right)}{\sqrt{a_{j_k}}\left(x_{j_k}^{(n_q)} - \frac{C}{\sqrt{a_{j_k}}}\right)} = 1$ then as in the proof of the part b) in I it follows that

$$\lim_{q\to\infty} \sqrt{a_{j_l}} x_{j_l}^{(n_q)} = \lim_{n_q\to\infty} \sqrt{a_{j_m}} x_{j_m}^{(n_q)} = \lim_{q\to\infty} \sqrt{a_{j_m}} x_{j_m}^{(n_q)} = C \pm \tau.$$

This contradicts to the relation $\lim_{q\to\infty} \left(x_{j_k}^{(n_q)} + x_{j_m}^{(n_q)} + x_{j_l}^{(n_q)} \right) = 1$.

The case $j_k = j_l$ and for the subsequence it hold $\sqrt{a_{j_l}}\left| x_{j_l}^{(n_q)} - \dfrac{C}{\sqrt{a_{j_l}}} \right| = \sqrt{a_{j_k}}\left| x_{j_k}^{(n_q)} - \dfrac{C}{\sqrt{a_{j_k}}} \right|,$ $\sqrt{a_{j_m}}\left| x_{j_m}^{(n_q)} - \dfrac{C}{\sqrt{a_{j_m}}} \right| = \sqrt{a_{j_k}}\left| x_{j_k}^{(n_q)} - \dfrac{C}{\sqrt{a_{j_k}}} \right|$ can be considered in a similar manner.

Thus we have that $\tau = 0$ and it follows that $\lim_{n\to\infty} \varphi\left(\mathbf{x}^{(n)}\right) = 0$, that is, $\lim_{n\to\infty} V^n(\mathbf{x}) = \mathbf{x}^*$.

The proof of Theorem 4.1 is complete. $\qquad\square$

If an operator has the property of being regular, then it satisfies the ergodic hypothesis, and by Theorem 4.1 non-Volterra QSO (5) is a regular transformation, so we have the following corollary.

Corollary 4.2. *For any* $\mathbf{x}^{(0)} \in S^2$*, there is the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k\left(\mathbf{x}^{(0)}\right)$$

that is, QSO (5) is an ergodic transformation.

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Bir novolterra kvadratik stoxastik operatorning regulyarligi
Rajabov Suyunjon

Mazkur maqolada ikki o'lchamli simpleksda aniqlangan bir novolterra kvadratik stoxastik operatorining dinamikasi o'rganilgan. Mazkur operatorning kamida to'rtta qo'zg'almas nuqtasi mavjudligi ko'rsatildi va qo'zg'almas nuqtalarning aniq ko'rinishlari topildi. Simpleks yoqlaridagi qo'zg'almas nuqtalarning nogiperbolik tipli va simpleks ichidagi yagona qo'zg'almas nuqtaning tortuvchi tipli bo'lishi isbotlandi. Parametrlarning ixtiyoriy qiymatlarida har qanday boshlang'ich nuqtaning trayektoriyasi yaqinlashishi isbotlandi.

Kalit so'zlar: Kvadratik stoxastik operator; Volterra operatori; novolterra operatori; regularity.

Регулярность одного невольтерровского квадратичного стохастического оператора
Ражабов Суюнжон

В данной статье исследуется динамика одного невольтерровского квадратичного стохастического оператора определенного в двумерном симплексе. Для этого оператора показано, что оператор имеет более чем четырех неподвижных точек и найдены явные виды всех неподвижных точек. Доказано что, все граничные неподвижные точки являются негиперболическими и единственная внутренняя неподвижная точка является аттрактором. А также доказано сходимость траектории произвольной начальной точки при любых значениях параметров.

Ключевые слова: квадратичный стохастический оператор; вольтерровский оператор; невольтерровский оператор; регулярность.

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