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On the Hölder Continuity of Weighted m-extremal Functions

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Abstract. In this paper, we prove that if the weighted m-subharmonic measure of compact K are Hölder continuous with respect to K, then it is Hölder continuous everywhere.

Keywords: m-subharmonic function, weighted m-Green function, weighted m-subharmonic measure.

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Dedicated to the memory of Professor Azimbay Sadullaev

1. Introduction and preliminaries

It is well known that, in pluripotential theory, the following important family of plurisubharmonic functions is referred to as the Lelong class

$$\mathcal{L} := \{u \in psh(\mathbb{C}^n) : u(z) \leqslant C_u + \ln^+ |z|,\ z \in \mathbb{C}^n\},$$

where $\ln^+ |z| = \max\{0, \ln |z|\}$ and C_u is a real constant depending on the function u.

Let $K \subset \mathbb{C}^n$ be a compact set. We define the function

$$V(z, K) = \sup\{u \in \mathcal{L} : u|_K \leqslant 0\}.$$

Then the upper regularization of this function,

$$V^*(z, K) = \overline{\lim_{w \to z}} V(w, K)$$

is called the *Green function* of the compact set K.

A set $E \subset \mathbb{C}^n$ is called *pluripolar* if there exists a plurisubharmonic function $\rho \in psh(\mathbb{C}^n)$ such that $\rho|_E = -\infty$ and $\rho \not\equiv -\infty$. If a compact set K is pluripolar, then its Green function satisfies $V^*(z, K) \equiv +\infty$. On the other hand, if K is not pluripolar, then $V^*(z, K) \in \mathcal{L}^+$, where

$$\mathcal{L}^+ := \{u \in \mathcal{L} : c_u + \ln^+ |z| \leqslant u(z),\ z \in \mathbb{C}^n\},$$

and c_u is a real constant depending on the function $u(z)$ (see, for example, [1]).

Definition 1.1. A point $z^0 \in K \subset \mathbb{C}^n$ is said to be *globally pluriregular* if $V^*(z^0, K) = 0$. If all points of a compact set K are globally pluriregular, then K is called a *globally pluriregular compact* set.

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In [2], it is proved that the global pluriregularity of a compact set K is equivalent to the continuity of its Green function, i.e., the function $V^*(z,K)$ is continuous in $\mathbb{C}^n$ if and only if the compact set K is globally pluriregular.

Definition 1.2. A compact set $K \subset \mathbb{C}^n$ is said to have the *Hölder continuity property* if there exist positive constants C and α such that

$$V(z,K) \leqslant C\,\delta^{\alpha} \quad \text{for all } z \in \mathbb{C}^n,\ \alpha \leqslant 1,\ 0<\delta\leqslant 1, \text{ and } \operatorname{dist}(z,K)\leqslant \delta. \tag{1.1}$$

It is not difficult to see that (1.1) implies K is globally pluriregular. However, the converse is not valid; that is, the global pluriregularity of the compact set K does not, in general, imply its Hölder continuity property (see, for example, [3]).

According to [3] (see also [4]), if K has Hölder continuity property, then its Green function $V^*(z,K)$ is Hölder continuous in $\mathbb{C}^n$,

$$|V^*(z',K) - V^*(z'',K)| \leqslant C\delta^{\alpha} \quad \forall\, z', z'' \in \mathbb{C}^n,\ |z'-z''| \leqslant \delta.$$

In this paper, we generalize this result to weighted harmonic measures and further we extend the results to the class of m-subharmonic functions. We begin by recalling some definitions before stating our main result.

Let $D \subset \mathbb{C}^n$ be a strongly m-regular domain (see Section 2 for definitions), K be a compact subset of D, and let $\psi(z)$ be a bounded and negative function defined on K. We denote by $\mathcal{U}(K,D,\psi)$ the class of all functions m-subharmonic functions $u(z)$ such that $u|_K \leqslant \psi|_K, u|_D < 0$ and define $\omega_m(z,K,D,\psi) = \sup\{u(z) : u \in \mathcal{U}(K,D,\psi)\}$. The upper regularization

$$\omega_m^*(z,K,D,\psi) = \overline{\lim_{w\to z}}\, \omega_m(w,K,D,\psi)$$

is called the *weighted (m,ψ)-subharmonic measure* ($\mathcal{P}_{(m,\psi)}$*-measure*) of the compact K with respect to D.

Theorem 1.3. *Let $\psi : K \to \mathbb{R}$ be a Hölder continuous function and $\omega_m^*(z,K,D,\psi)$ be as above. Assume that there exists $C>0$ and $0<\alpha\leqslant 1$ such that for any $z \in D$ we have*

$$|\omega_m^*(z,K,D,\psi) - \psi(w)| \leqslant C \cdot |z-w|^{\alpha}, \tag{1.2}$$

where $w \in K$ is a point closest to z, i.e., $|z-w| = \operatorname{dist}(z,K)$. Then $\omega_m^*(z,K,D,\psi)$ is Hölder continuous in D.

Theorem 1.3 states that the Hölder continuity of $\omega_m^*(z,K,D,\psi)$ with respect to the compact set K implies its Hölder continuity everywhere. It is important to note that the Hölder continuity of ψ is essential; without this assumption, one cannot, in general, obtain the Hölder continuity of $\omega_m^*(z,K,D,\psi)$ (see Example 4.3). Moreover, as a generalization of [3], condition (1.2) replaces the condition (1.1).

This paper organized as follows. In Section 2, we provide the definition of m-subharmonic functions and some necessary notions. In Section 3, we establish a sufficient condition for the Hölder continuity of the weighted m-Green function of the compact set K, assuming that the weight function ψ is Hölder continuous on K. Finally, in Section 4, we prove Theorem 1.3.

2. m-subharmonic functions

The class of m-subharmonic functions generalizes the class of plurisubharmonic functions, which are defined by condition $dd^c u \geqslant 0$ (see, for example, [1,5–8]), where $d = \partial + \bar\partial$, $d^c = \dfrac{\partial - \bar\partial}{4i}$. The m-subharmonic functions in a domain $D \subset \mathbb{C}^n$ is defined using the operators

$$(dd^c u)^k \wedge \beta^{n-k}, \quad 1 \leqslant k \leqslant n, \tag{2.1}$$

where $\beta = dd^c|z|^2$ are the standard canonical (1,1) forms in $\mathbb{C}^n$. The operator (2.1) gives the Laplace operator for $k = 1$ and the Monge–Ampere operator for $k = n$. The operator (2.1) is called the complex Hessian operator.

Definition 2.1. A twice differentiable function $u(z) \in C^2(D)$ is called *m-subharmonic* if the following conditions

$$(dd^c u)^k \wedge \beta^{n-k} \geqslant 0 \quad \text{for all } k = 1, 2, \dots, n-m+1$$

hold at each point $z^0 \in D$.

Z. Blocki proved that for all twice differentiable m-subharmonic functions $u,\ v_1,\ v_2, \dots,\ v_{n-m}$, the following inequality holds

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geqslant 0. \tag{2.2}$$

On the other hand, if a twice differentiable function u satisfies (2.2) for all twice differentiable m-subharmonic functions $v_1,\ v_2,\ \dots,\ v_{n-m}$, then u is necessarily m-subharmonic (see [9]). Using this, Abdullaev and Sadullaev defined m-subharmonic functions in the class of the upper semicontinuous functions (see [10]).

Definition 2.2. A function $u \in L^1_{\text{loc}}(D)$ is called *m-subharmonic* in the domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice differentiable m-subharmonic functions $v_1, v_2, \dots, v_{n-m}$, the current

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$$

is positive, i.e., for any positive test function $\omega \in F^{0,0}(D)$ we have

$$\int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geqslant 0.$$

The class of m-subharmonic functions is denoted by $sh_m(D)$. It is clear that

$$psh = sh_1 \subset sh_2 \subset sh_m \subset \dots \subset sh_n = sh. \tag{2.3}$$

Definition 2.3. A set $E \subset D$ is called *m-polar* in $D \subset \mathbb{C}^n$ if there exists a function $u \in sh_m(D)$, $u \not\equiv -\infty$, such that $u|_E = -\infty$.

It follows from (2.3) that an m-polar set is polar in the sense of classical potential theory.

Definition 2.4. A domain $D \subset \mathbb{C}^n$ is called *m-regular* if there exists a function $\rho \in sh_m(D)$ such that $\rho|_D < 0$, $\lim\limits_{z \to \partial D} \rho(z) = 0$, i.e., $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. It is called *strongly m-regular* if $\rho \in sh_m(D^+) \cap C^2(D^+)$ and is strictly m-subharmonic in D^+, where D^+ is a neighborhood of the closure $\bar{D}$.

m−Green function. It is known that in the class of m-subharmonic functions, the Lelong class $\mathcal{L}_m$ is defined as follows (see [11])

$$\mathcal{L}_m := \{u \in sh_m(\mathbb{C}^n) : u(z) < 1,\ z \in \mathbb{C}^n\}, \quad 1 < m \leqslant n,$$

and set $\mathcal{L}_1 := \mathcal{L}$. Let $K \subset \mathbb{C}^n$ be a compact set. We define the function

$$V_m(z, K) = \sup\{u \in \mathcal{L}_m : u|_K \leqslant 0\}.$$

The upper regularization of this function,

$$V^*_m(z, K) = \varlimsup_{w \to z} V_m(w, K)$$

is called the $m-$*Green function* of the compact K.

Since $\mathcal{L} = \mathcal{L}_1$, the $1-$Green function $V_1^*(z, K)$ of a compact set K coincides with the classical Green function $V^*(z, K)$, i.e.,

$$V_1^*(z, K) = V^*(z, K).$$

Unlike the case $m = 1$ for $1 < m \leqslant n$, if K is m-polar, we have $V_m^*(z, K) \equiv 1$. Moreover, if K is not m-polar, then $V_m^*(z, K) \in \mathcal{L}_m$ (see [11]).

3. Hölder continuity of a weighted m−Green function

Let $K \subset \mathbb{C}^n$ be a compact set and let $\psi(z)$ be a bounded, real-valued function defined on it. Consider the class of functions constructed using the Lelong class and $\psi(z)$

$$\mathcal{L}_m(K, \psi) := \{u \in \mathcal{L}_m : u|_K \leqslant \psi\}, \quad 1 \leqslant m \leqslant n.$$

Note that for $u \in \mathcal{L}_m$, we have $u < 1$ when $1 < m \leqslant n$. Therefore, in this case, we can assume that $\psi(z) < 1$. Define

$$V_m(z, K, \psi) = \sup\{u(z) : u \in \mathcal{L}_m(K, \psi)\}, \quad 1 \leqslant m \leqslant n, \ z \in \mathbb{C}^n.$$

Definition 3.1. The following function

$$V_m^*(z, K, \psi) = \overline{\lim_{w \to z}} V_m(w, K, \psi),$$

is called the *weighted* $m-$*Green function* of the compact set K with respect to the weight function $\psi(z)$.

Note that when $\psi(z) \equiv 0$, the function $V_m^*(z, K, \psi)$ coincides with the $m-$Green function $V_m^*(z, K)$, i.e., $V_m^*(z, K, 0) \equiv V_m^*(z, K)$.

The relationship between the m-Green function and the weighted $m-$Green function of K with respect to the weight function ψ is characterized by the following inequalities (see [12, 13] and [11]). For $m = 1$, we have

$$\inf_{z \in K} \psi(z) + V_1^*(z, K) \leqslant V_1^*(z, K, \psi) \leqslant \sup_{z \in K} \psi(z) + V_1^*(z, K) \tag{3.1}$$

and for $1 < m \leqslant n$, we have

$$\inf_{z \in K} \psi(z) + \big(1 - \inf_{z \in K} \psi(z)\big) \cdot V_m^*(z, K) \leqslant V_m^*(z, K, \psi) \leqslant \sup_{z \in K} \psi(z) + \big(1 - \sup_{z \in K} \psi(z)\big) \cdot V_m^*(z, K). \tag{3.2}$$

From (3.1) and (3.2), it is easy to see that if K is m-polar, then its weighted $m-$Green function satisfies

$$V_1^*(z, K, \psi) \equiv +\infty;$$

and

$$V_m^*(z, K, \psi) \equiv 1,$$

for $1 < m \leqslant n$. If K is not m-polar, then $V_m^*(z, K, \psi) \in \mathcal{L}_m$. The following theorem is the main result of this section.

Theorem 3.2. *Let $K \subset \mathbb{C}^n$ be a compact set and $\psi : K \to \mathbb{R}$ be a Hölder continuous function. Assume that, there exists $C > 0$ and $0 < \alpha \leqslant 1$ such that for any $z \in \mathbb{C}^n$ we have*

$$|V_m(z, K, \psi) - \psi(w)| \leqslant C \cdot |z - w|^\alpha,$$

where $w \in K$ is a point closest to z, i.e., $|z - w| = \operatorname{dist}(z, K)$. Then the function $V_m(z, K, \psi)$ is Hölder continuous in $\mathbb{C}^n$.

Proof. From the assumption of the theorem, it follows that

$$V_m^*(z^0, K, \psi) = \psi(z^0) \quad \forall\, z^0 \in K,$$

which implies that the weighted $m-$Green function of K is continuous (see [13] and [11]) and

$$V_m^*(z, K, \psi) = V_m(z, K, \psi) \quad \forall z \in \mathbb{C}^n.$$

Since $\psi(z)$ is Hölder continuous on K, there exist constants $C_1 > 0$ and $0 < \alpha_1 \leqslant 1$ such that for all $z',\ z'' \in K$,

$$|\psi(z') - \psi(z'')| \leqslant C_1 \delta^{\alpha_1}.$$

Fix an arbitrary $t \in \mathbb{C}^n$ such that $|t| \leqslant \delta < 1$. Then $V_m^*(z+t, K, \psi) \in \mathcal{L}_m$. Let $w' \in K$ be a point such that $\operatorname{dist}(z+t, K) = |z+t-w'|$. If $z \in K$, then we have $|z+t-w'| \leqslant \delta$, and by the triangle inequality, $|z - w'| \leqslant 2\delta$, since $|t| \leqslant \delta$. Then, we obtain the following inequality for all $z \in K$

$$|V_m(z+t, K, \psi) - \psi(z)| \leqslant |V_m(z+t, K, \psi) - \psi(w')| + |\psi(z) - \psi(w')| \leqslant C_2 \delta^{\alpha_2},$$

where $C_2 = 3\max\{C, C_1\}$ and $\alpha_2 = \min\{\alpha, \alpha_1\}$. Hence,

$$V_m(z+t, K, \psi) - C_2\delta^{\alpha_2} \leqslant \psi(z) \quad \forall z \in K,$$

and thus,

$$V_m(z+t, K, \psi) - C_2\delta^{\alpha_2} \in \mathcal{L}_m(K, \psi).$$

Consequently, by the definition of $V_m(z, K, \psi)$ we have

$$V_m(z+t, K, \psi) \leqslant C_2\delta^{\alpha_2} + V_m(z, K, \psi) \quad \forall z \in \mathbb{C}^n$$

and

$$V_m(z+t, K, \psi) - V_m(z, K, \psi) \leqslant C_2\delta^{\alpha_2} \quad \forall\, z \in \mathbb{C}^n. \tag{3.3}$$

Now we will prove

$$-C_2\delta^{\alpha_2} \leqslant V_m(z+t, K, \psi) - V_m(z, K, \psi), \quad |t| \leqslant \delta \quad \forall z \in \mathbb{C}^n.$$

If $z + t \in K$, then $|z - w| \leqslant \delta$ and $|z + t - w| \leqslant 2\delta$. Recall that $w \in K$ is chosen so that $|z - w| = \operatorname{dist}(z, K)$. Then we have the following inequality

$$|V_m(z, K, \psi) - \psi(z+t)| \leqslant |V_m(z, K, \psi) - \psi(w)| + |\psi(z+t) - \psi(w)| \leqslant C_2\delta^{\alpha_2}.$$

Therefore,

$$V_m(z, K, \psi) - C_2\delta^{\alpha_2} \leqslant \psi(z+t) \quad \forall z+t \in K,$$

and thus,

$$V_m(z, K, \psi) - V_m(z+t, K, \psi) \leqslant C_2\delta^{\alpha_2} \quad \forall z \in \mathbb{C}^n. \tag{3.4}$$

From inequalities (3.3) and (3.4), it follows that the function $V_m(z, K, \psi)$ is Hölder continuous in $\mathbb{C}^n$. □

Theorem 3.2 yields the following corollary and generalizes Proposition 3.5 in [3] for m-Green function.

Corollary 3.3. *Let $K \subset \mathbb{C}^n$ be a compact set. Assume that, there exists $C > 0$ and $0 < \alpha \leqslant 1$ such that for any $z \in \mathbb{C}^n$, we have*

$$V_m(z, K) \leqslant C \cdot |z - w|^{\alpha},$$

where $w \in K$ is a point closest to z, i.e., $|z - w| = \operatorname{dist}(z, K)$. Then the function $V_m(z, K)$ is Hölder continuous in $\mathbb{C}^n$.

4. Hölder continuity of the weighted m-subharmonic measure

In this section we prove Theorem 1.3. Let us first recall some necessary definitions. Let $D \subset \mathbb{C}^n$ be a strongly m-regular domain, E be a subset of D and let $\psi(z)$ be a bounded and negative function defined on E.

We denote by $\mathcal{U}(E, D, \psi)$ the class of all functions $u(z) \in sh_m(D)$ such that

$$u|_E \leqslant \psi|_E, \quad u|_D < 0$$

and define

$$\omega_m(z, E, D, \psi) = \sup\{u(z) : u \in \mathcal{U}(E, D, \psi)\}.$$

Definition 4.1. The upper regularization

$$\omega_m^*(z, E, D, \psi) = \overline{\lim_{w \to z}} \, \omega_m(w, E, D, \psi)$$

is called the *weighted* (m, ψ)*-subharmonic measure* ($\mathcal{P}_{(m,\psi)}$*-measure*) of the set E with respect to D.

Note that $\omega_m^*(z, E, D, -1)$ (i.e., $\psi \equiv -1$) coincides with the classical m-subharmonic measure for functions $u(z) \in sh_m(D)$, i.e.,

$$\omega_m^*(z, E, D, -1) = \omega_m^*(z, E, D).$$

As seen from Definition 4.1, the function $\omega_m^*(z, E, D, \psi)$ is m-subharmonic in D. The weighted (m, ψ)-subharmonic measure satisfies the properties of the classical m-subharmonic measure (see [14]).

From now on we fix a compact set $K \subset D$.

Definition 4.2. A point $z^0 \in K$ is said to be *globally* (m, ψ)*-regular* if $\omega_m^*(z^0, K, D, \psi) = \psi(z^0)$. If all points of the compact set K are globally (m, ψ)-regular, then K is called a *globally* (m, ψ)*-regular compact.*

If $\psi(z)$ is continuous on the compact set K, then the global ψ-regularity implies that the weighted (m, ψ)-subharmonic measure is continuous in D (see [14]). We are now ready to prove our main result.

Proof of Theorem 1.3. Since D is a strongly m-regular domain, there exists a neighborhood D^+ of $\bar{D}$ and a function $\rho \in sh_m(D^+) \cap C^2(D^+)$ such that

$$D = \{z \in D^+ : \rho(z) < 0\}, \quad \lim_{z \to \partial D} \rho(z) = 0,$$

Furthermore, given that the function ψ is negative on the compact set K, the function $\omega_m^*(z, E, D, \psi)$ can be extended as an m-subharmonic function to the domain D^+, i.e., the following function

$$\tilde{\omega}^*(z) := \begin{cases} \omega_m^*(z, E, D, \psi), & z \in D, \\ \rho(z), & z \in D^+ \setminus D, \end{cases}$$

defines an m-subharmonic function on D^+ (see [14]). Assumption (1.2) implies that K is (m, ψ)-regular. Consequently,

$$\omega_m^*(z, K, D, \psi) = \omega_m(z, K, D, \psi) \quad \forall z \in D,$$

and $\tilde{\omega}(z) = \tilde{\omega}^*(z)$, for all $z \in D^+$, where

$$\tilde{\omega}(z) = \begin{cases} \omega_m(z, K, D, \psi), & z \in D, \\ \rho(z), & z \in D^+ \setminus D. \end{cases}$$

So $\tilde{\omega}(z)$ is m-subharmonic in D^+. By its definition, it is enough to show that $\tilde{\omega}(z)$ is Hölder continuous on D.

Since $\rho \in C^2(D^+)$, it satisfies the Lipschitz condition in any relatively compact subdomain $G \subset\subset D^+$, i.e.,

$$\exists L > 0,\ L = L(G), \quad |\rho(z') - \rho(z'')| \leqslant L|z' - z''| \quad \forall\, z', z'' \in G. \tag{4.1}$$

Fix G such that $D \subset\subset G \subset\subset D^+$ and δ sufficiently small so that

$$0 < \delta < \min\{\operatorname{dist}(K, \partial D),\ \operatorname{dist}(D, \partial G),\ 1\}.$$

It is enough to prove that there exist constants $A > 0$ and $\alpha \in (0, 1]$ such that

$$|\omega_m(z', K, D, \psi) - \omega_m(z'', K, D, \psi)| \leqslant A|z' - z''|^\alpha$$

for any $z', z'' \in D$ with $|z' - z''| \leqslant \delta$. By Definition 4.1, the function $\omega_m(z, K, D, \psi)$ is bounded and satisfies

$$\inf_{z \in K} \psi(z) \leqslant \omega_m(z, K, D, \psi) \leqslant 0 \quad \forall z \in D.$$

As $\psi(z)$ is Hölder continuous on K, there exist constants $C_1 > 0$ and $0 < \alpha_1 \leqslant 1$ such that for all $z', z'' \in K$, we have

$$|\psi(z') - \psi(z'')| \leqslant C_1 \delta^{\alpha_1}.$$

Fix an arbitrary $t \in \mathbb{C}^n$ such that $|t| \leqslant \delta$. Then $\tilde{\omega}(z+t)$ is m-subharmonic for every point $z \in D$. Let $w' \in K$ be a point such that $\operatorname{dist}(z + t, K) = |z + t - w'|$. If $z \in K$, then $|z + t - w'| \leqslant \delta$, and by the triangle inequality, $|z - w'| \leqslant 2\delta$. Note that by definition of δ and t for $z \in K$, we have $z + t \in D$. Then, for all $z \in K$, we have the following inequality

$$|\omega_m(z + t, K, D, \psi) - \psi(z)| \leqslant |\omega_m(z + t, K, D, \psi) - \psi(w')| + |\psi(z) - \psi(w')| \leqslant C_2 \delta^{\alpha_2},$$

where $C_2 = 3 \max\{C, C_1\}$ and $\alpha_2 = \min\{\alpha, \alpha_1\}$. Therefore, for all $z \in K$, we have

$$\omega_m(z + t, K, \psi) - C_2 \delta^{\alpha_2} \leqslant \psi(z).$$

Thanks to (4.1), the condition $|t| \leqslant \delta$, and $\rho|_{\partial D} = 0$, for $z+t \notin D$, we have $\tilde{\omega}(z+t) = \rho(z+t) \leqslant L\delta$. Thus,

$$\tilde{\omega}(z + t) - C_3 \delta^{\alpha_2} \in \mathcal{U}(E, D, \psi),$$

where $C_3 = \max\{L,\ C_2\}$. Hence, we obtain the following estimate

$$\tilde{\omega}(z + t) - C_3 \delta^{\alpha_2} \leqslant \omega_m(z, K, D, \psi) \quad \forall z \in D$$

and

$$\tilde{\omega}(z + t) - \omega_m(z, K, \psi, D) \leqslant C_3 \delta^{\alpha_2} \quad \forall z \in D.$$

Consequently, for $z \in D$ with $z + t \in D$, we have

$$\omega_m(z + t, K, \psi, D) - \omega_m(z, K, \psi, D) \leqslant C_3 \delta^{\alpha_2} \quad \forall z \in D. \tag{4.2}$$

Now we will prove

$$-C_3 \delta^{\alpha_2} \leqslant \omega_m(z + t, K, D, \psi) - \omega_m(z, K, D, \psi), \quad |t| \leqslant \delta \quad \forall z \in D.$$

Recall that $w \in K$ such that $|z - w| = \operatorname{dist}(z, K)$. If $z + t \in K$, then we have $|z - w| \leqslant \delta$ and by triangle inequality $|z + t - w| \leqslant 2\delta$. Then for all $z \in D$ with $z + t \in K$, we obtain the following estimate

$$|\omega_m(z, K, D, \psi) - \psi(z + t)| \leqslant |\omega_m(z, K, D, \psi) - \psi(w)| + |\psi(z + t) - \psi(w)| \leqslant C_2\delta^{\alpha_2}.$$

Thus,

$$\omega_m(z, K, \psi) - C_2\delta^{\alpha_2} \leqslant \psi(z + t) \quad \forall\, z + t \in K.$$

Consequently,

$$\tilde{\omega}(z) - C_3\delta^{\alpha_2} \leqslant \omega_m(z + t, K, \psi) \quad \forall\, z + t \in D.$$

Therefore,

$$\tilde{\omega}(z) - \omega_m(z + t, K, D, \psi) \leqslant C_3\delta^{\alpha_2} \quad \forall\, z + t \in D.$$

Hence, for $z \in D$ with $z + t \in D$, we have

$$\omega_m(z, K, D, \psi) - \omega_m(z + t, K, D, \psi) \leqslant C_3\delta^{\alpha_2}. \tag{4.3}$$

From (4.2) and (4.3), it follows that $\omega_m(z, K, D, \psi)$ is Hölder continuous in D. □

In Theorem 1.3, the Hölder continuity of the function $\psi(z)$ on the compact set K plays a crucial role. Without this condition, a priori the conclusion of the theorem does not hold, i.e., the function $\omega_m^*(z, K, D, \psi)$ is not Hölder continuous in the domain D. An example of such a case is presented below.

Example 4.3. Let $K = \overline{B\left(0, \frac{1}{4}\right)}$ be the closed ball centered at 0 with radius $\dfrac{1}{4}$ and let $D = B\left(0, \frac{1}{2}\right) \subset \mathbb{C}$ be the open ball of radius $\dfrac{1}{2}$. Define the following function

$$\psi(z) = \begin{cases} -\dfrac{1}{\ln|z|} - 2, & |z| \neq 0, \\ -2, & |z| = 0. \end{cases}$$

It is easy to verify that

$$\psi(z) \in sh(D) \cap C(D)$$

and that the compact set K is locally regular, i.e., for every $z \in K$ and any neighborhood $V \ni z$, the following holds

$$\omega^*(z, K \cap \bar{V}, D) = -1.$$

Then, by [14, Theorem 2], the compact set K is locally ψ-regular, i.e., for every $z \in K$ and any neighborhood V of z, the following holds

$$\omega^*(z, K \cap \bar{V}, D, \psi) = \psi(z).$$

As a result,

$$\omega^*(z, K, D, \psi) = \psi(z) \quad \forall z \in K.$$

Consequently, although the function $\omega_m(z, K, D, \psi)$ is continuous and subharmonic in D, it is not Hölder continuous because $\psi(z)$ does not satisfy the Hölder condition in any neighborhood of the point $z = 0$ for any $\alpha \in (0, 1]$.

Theorem 1.3 yields the following corollaries.

Corollary 4.4. *Let $K \subset \mathbb{C}^n$ be a compact set. Assume that, there exists $C > 0$ and $0 < \alpha \leqslant 1$ such that for any $z \in D$, we have*

$$1 + \omega_m(z, K, D) \leqslant C|z - w|^\alpha,$$

where $w \in K$ is a point closest to z, i.e., $|z - w| = \mathrm{dist}(z, K)$. Then the function $\omega_m(z, K, D)$ is Hölder continuous in the domain D.

Corollary 4.5. *If the weighted (m, ψ)-subharmonic measure $\omega^*_m(z, K, D, \psi)$ is Hölder continuous in the domain D, then it can be extended as an m-subharmonic and Hölder continuous function to some neighborhood of $\bar{D}$.*

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О непрерывности по Гёльдеру взвешенных m-экстремальных функций

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Аннотация. В данной работе мы доказываем, что если взвешенная m-субгармоническая мера компакта K является непрерывной по Гёльдеру относительно K, то она непрерывна по Гёльдеру всюду.

Ключевые слова: m-субгармоническая функция, взвешенная m-функция Грина, взвешенная m-субгармоническая мера.