Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences

Volume 6 | Issue 2 Article 2

6-15-2023

Weighted m-subharmonic measure and (m, ⯑)-regularity of compacts

Kobiljon Kuldoshev
National University of Uzbekistan, Tashkent, Uzbekistan, qobiljonmath@gmail.com

Nurbek Narzillaev
National University of Uzbekistan, Tashkent, Uzbekistan, n.narzillaev@nuu.uz

Recommended Citation

Kuldoshev, Kobiljon and Narzillaev, Nurbek (2023) "Weighted m-subharmonic measure and (m, ⯑)-regularity of compacts," *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*: Vol. 6: Iss. 2, Article 2.
DOI: https://doi.org/10.56017/2181-1318.1242

WEIGHTED $m-$SUBHARMONIC MEASURE AND $(m,\psi)-$REGULARITY OF COMPACTS

Kuldoshev K., Narzillaev N.
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: qobiljonmath@gmail.com, n.narzillaev@nuu.uz

Abstract

It is known that, the $m-$subharmonic measure of a set $E \subset D,$ related to a domain $D \subset \mathbb{C}^n$, is defined by $m-$subharmonic functions in D. In this article we define a generalization of the $m-$subharmonic measures and prove some of their properties.

Keywords: *$m-$subharmonic function, $m-$subharmonic measure, weighted $m-$subharmonic measure, $m-$polar set, $m-$regular compact, $(m,\psi)-$regular compact.*

Mathematics Subject Classification (2010): *35R02, 35K05, 35A22.*

Introduction

The pluripotential theory is based on plurisubharmonic (psh) functions and it is related to the Monge–Amper operator $(dd^c u)^n$, here as usual $d = \partial + \bar{\partial}$, $d^c = \frac{\partial - \bar{\partial}}{4i}$, $\partial = \frac{\partial}{\partial z_1} dz_1 + \frac{\partial}{\partial z_2} dz_2 + ... + \frac{\partial}{\partial z_n} dz_n$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial}{\partial \bar{z}_2} d\bar{z}_2 + ... + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n$ and $dd^c = \frac{i}{2}\partial\bar{\partial}$. This theory is based on researches in numerous fundamental works of E. Bedford, A. Taylor [6], A. Sadullaev [2, 3, 4], J. Siciak [7] and others.

The theory of $m-$subharmonic (sh_m) functions plays an important role in the potential theory. It expands and develops the pluripotential theory, which is the main subject for studying analytic functions of several complex variables and plurisubharmonic functions.

The sh_m functions are related to the operators

$$(dd^c u)^m \wedge \beta^{n-m}, \qquad 1 \le m \le n, \tag{1}$$

where $\beta = dd^c |z|^2 = \frac{i}{2} \sum\limits_{i=1}^{n} dz_i \wedge d\bar{z}_i$ is canonical (1,1) form in $\mathbb{C}^n$.

Then $dV_n = \frac{1}{n!}\beta^n$ is the volume form in $\mathbb{C}^n$. Since $(dd^c u) \wedge \beta^{n-1} = \Delta u \beta^n$ and $(dd^c u)^m = \left(u_{j,\bar{k}}\right) dV_n = \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) dV_n$, then operator (1) gives the Laplase operator for $m = 1$ and the Monge-Ampere operator for $m = n$. The operator (1) is called the complex operator in Hessians, as it can be shown that $(dd^c u)^m \wedge \beta^{n-m} = m!\,(n-m)! H_m(u)\,\beta^n$, where $H_m(u) = \sum\limits_{1 \le j_1 < j_2 < ... < j_m \le n} \lambda_{j_1} \cdot \lambda_{j_2} \cdot ... \cdot \lambda_{j_m} -$ is the Hessian of dimension m of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ of the matrix $\left(u_{j,\bar{k}}\right)$. The Potential theory related to the class of sh_m functions was constructed in the works [1, 2, 5]. In their studies, the class of sh_m functions was defined in the class of integrable functions $L^1_{loc}(D)$.

Definition 1. *Let $u \in C^2(D)$, where $D \subset \mathbb{C}^n$, is called $m-$ subharmonic $(1 \le m \le n)$ at the point $z^0 \in D$, if the eigenvalues $\lambda(u) = (\lambda_1(u), \lambda_2(u), ..., \lambda_n(u))$ of the matrix $\left(u_{j\bar{k}}\right)\big|_{z=z^0}$ belong to $\Gamma_m = \{\lambda : H_1(\lambda) \ge 0,\ H_2(\lambda) \ge 0,\ ...,\ H_m(\lambda) \ge 0\}$. A function $u \in C^2(D)$ is called $m-$ subharmonic in D if it is $m-$ subharmonic at every point of $z^0 \in D$.*

In other words, a function $u \in C^2(D)$ is called $m-$subharmonic if the conditions $(dd^c u)^k \wedge \beta^{n-k} \ge 0$, $\forall k = 1, 2, ..., m$, holds.

It is known that for all twice differentiable $m-$ subharmonic functions $u, v_1, v_2, ..., v_{m-1}$ it is true

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m} \ge 0. \tag{2}$$

Moreover, if a twice differentiable function u satisfies (2) for all twice differentiable $m-$subharmonic functions $v_1, v_2, \dots, v_{m-1}$ then u is necessarily $m-$subharmonic function. Using this, we can define $m-$subharmonic functions in the class $L^1_{loc}(D)$.

Definition 2. *A function $u \in L^1_{loc}(D)$ is called $m-$subharmonic in a domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice differentiable $m-$subharmonic functions $v_1, v_2, \dots, v_{m-1}$ the current $dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}$ defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}\right](\omega) =$$

$$= \int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m} \wedge dd^c \omega, \quad \omega \in F^{(0,0)} \textit{ is positive,}$$

$$\int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m} \wedge dd^c \omega \ge 0, \quad \forall \omega \ge 0.$$

Definition 3. *A set $E \subset D$ is called $m-$polar in D, if there exist a function $u(z) \in sh_m(D)$, $u(z) \not\equiv -\infty$, such that $u|_E = -\infty$.*

The main objects of pluripotential theory are a plurisubharmonic measure ($\mathcal{P}-$ measure), a Green's function and a condenser capacity for plurisubharmonic functions (see [4, 6, 7, 8]). Nowadays, in connection with integral estimates of polynomials in approximation questions, the plurisubharmonic measure and the Green's function with a certain weight function have been investigated in the works [10, 11, 12]. The aim of this work is to introduce a $m-$subharmonic measure ($\mathcal{P}_m-$ measure) with a weighted function ψ.

The $m-$ subharmonic measure is defined as an extremal function in the class of $m-$subharmonic (sh_m) functions. Let $E \subset D$ be some subset of the domain $D \subset \mathbb{C}^n$. For simplicity, we assume that D is strongly $m-$convex, i.e. $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, where $\rho(z)$ is a continuous and $m-$subharmonic function in some neighborhood G, of $\overline{D}$, $G \supset \overline{D}$. We denote by $\mathcal{U}(E, D)$ the class of all functions $u \in sh_m(D)$, such that $u|_E \le -1$, $u|_D < 0$ and let $\omega(z, E, D) = \sup\{u(z) : u(z) \in \mathcal{U}(E, D)\}$.

Definition 4. *The regularization*

$$\omega^*(z, E, D) = \overline{\lim_{w \to z}}\, \omega(w, E, D) = \lim_{\varepsilon \to 0} \sup_{w \in B(z,\varepsilon)} \omega(w, E, D)$$

is called the $m-$ *subharmonic measure* ($\mathcal{P}_m-$ *measure) of* E *with respect to* D *(see [1, 2]).*

Let $D \subset \mathbb{C}^n$ be a domain and $K \subset D$ a compact.

Definition 5. *A point* $z^0 \in K$ *is said to be globally* $m-$ *regular if* $\omega^*(z^0, K, D) = -1$. *It is said to be locally* $m-$ *regular if for any neighborhood* $B, z^0 \in B \subset \mathbb{C}^n$, *the intersection* $K \cap \overline{B}$ *is globally* $m-$ *regular at the point* z^0, *i.e.* $\omega^*\left(z^0, K \cap \overline{B}, D\right) = -1$. *If all points of the compact* K *are* $m-$ *regular, then* K *is called a* $m-$ *regular compact [1, 2].*

The following work uses Hartogs and Choquet's lemmas.

Lemma 1 (Hartogs', see [2]). *Suppose that* $g(z)$ *is a continuous real valued function in a domain* $D \subset \mathbb{C}^n$ *and* $u_j(z)$, $j \in \mathbb{N}$, *is a sequence of locally uniformly upper bounded subharmonic functions such that*

$$\overline{\lim_{j\to\infty}} u_j(z) \le g(z)$$

at each point $z \in D$. *Then for any compact set* $K \subset D$, *for any* $\varepsilon > 0$ *there exists an integer* j_0 *such that* $u_j(z) \le g(z) + \varepsilon$, *for each* $z \in K$ *and each* $j > j_0$.

Let u_α be a family of upper semicontinuous functions in $D \subset \mathbb{C}^n$ which is locally uniformly bounded from above. Then the upper envelope $u(z) = \sup\limits_{\alpha} u_\alpha(z)$ is not always upper semicontinuous. But if we consider we define the upper semicontinuous regularization $u^*(z) = \lim\limits_{\varepsilon\to 0} \sup\limits_{w\in B(z,\varepsilon)} u(w)$, where $B(z, \varepsilon) \subset D$ is a ball. It is easy to check that u^* is upper semicontinuous.

Lemma 2 (Choquet, see [2, 9]). *Every family* $\{u_\alpha\}$ *has a countable subfamily* $\{u_{\alpha_j}\}$ *whose upper envelope* $v = \sup\limits_j \{u_{\alpha_j}\}$ *satisfies the conditions* $v \le u \le u^* = v^*$.

Theorem 1 (see [2]). *Let* $\{u_\alpha(z)\}$, $\alpha \in \Lambda$, *be an arbitrary locally uniformly upper bounded family of* $m-$*subharmonic functions in the domain* $D \subset \mathbb{C}^n$ *and* $u(z) = \sup\limits_\alpha \{u_\alpha(z)\}$. *Then the regularization* $u^*(z)$ *of* $u(z)$ *is an* $m-$*subharmonic function in* D.

1 Weighted $m-$subharmonic measure and its properties

Let $D \subset \mathbb{C}^n$ be a strongly $m-$convex domain, $E \subset D$ be any fixed set and $\psi(z)$ be bounded and negative function in E. We denote by $\mathcal{U}(E, D, \psi)$ the class of all functions $u(z) \in sh_m(D)$, such that $u|_E \le \psi|_E$, $u|_D < 0$ and let

$$\omega(z, E, D, \psi) = \sup\{u(z) : u(z) \in \mathcal{U}(E, D, \psi)\}.$$

Definition 6. *The function*

$$\omega^*(z,\, E,\, D,\, \psi) = \overline{\lim_{w\to z}}\, \omega(w, E, D, \psi)$$

is called a $(m,\psi)-$subharmonic measure ($\mathcal{P}_{m,\psi}-$measure) of E with respect to D.

Note that $\omega^*(z,\, E,\, D,\, -1)$ coincides with the $\mathcal{P}_m$ measure in the classical potential theory, i.e. $\omega^*(z, E, D, -1) = \omega^*(z, E, D)$. According to the theorem 1 the function $\omega^*(z,\, E,\, D,\, \psi)$ *is* $m-$subharmonic and possesses many properties of the $\omega^*(z,\, E,\, D)$.

Below we list some properties of the $\mathcal{P}_{m,\,\psi}-$measure.

Proposition 1. (monotonicity)

a) let $\psi_1|_E \le \psi_2|_E$. Then $\omega^*(z,\, E,\, D,\, \psi_1) \le \omega^*(z,\, E,\, D,\, \psi_2)$ for all $z \in D$.

b) let $E_1 \subset E_2 \subset D_1 \subset D_2$. Then
$\omega^*(z,\, E_2,\, D_2,\, \psi) \le \omega^*(z,\, E_1,\, D_2,\, \psi) \le \omega^*(z,\, E_1,\, D_1,\, \psi)$ for all $z \in D_1$.

Proposition 2 (On two constants Theorem)**.** *If the function $u(z)$ is $m-$subharmonic in the domain $D \subset \mathbb{C}^n$ and $u|_D < C, u|_E \le c$, where $E \subset D$, then the inequality*

$$u(z) \le C \cdot \left(1 - \frac{\omega^*(z, E, D, \psi)}{\inf\limits_{z\in E} \psi(z)}\right) + c \cdot \frac{\omega^*(z, E, D, \psi)}{\inf\limits_{z\in E} \psi(z)}$$

holds for all $z \in D$.

Proposition 3. *Let $E = \bigcup\limits_{j=1}^{\infty} E_j,\ \ E_j \subset D$. Then for all $z \in D$ the inequality $\omega^*(z,\, E,\, D,\, \psi) \ge \sum\limits_{j=1}^{\infty} \omega^*(z,\, E_j,\, D,\, \psi)$ holds.*

Proof. Take $\forall u_j \in \mathcal{U}(E_j,\, D,\, \psi)$ and consider the class

$$\left\{\sum_{j=1}^{\infty} u_j(z) :\ u_j \in \mathcal{U}(E_j,\, D,\, \psi)\right\}.$$

Since u_j is $m-$subharmonic and negative in D, it follows that the sum $\sum\limits_{j=1}^{\infty} u_j(z)$ is also $m-$subharmonic function in D. We can easily check that $\sum\limits_{j=1}^{\infty} u_j(z) \in \mathcal{U}(E,\ D,\ \psi)$.

Hence, $\left\{\sum\limits_{j=1}^{\infty} u_j(z) :\ u_j \in \mathcal{U}(E_j,\, D,\ \psi)\right\} \subset \mathcal{U}(E,\ D,\ \psi)$ and

$$\omega(z,\, E,\, D,\, \psi) \ge \sup\left\{\sum_{j=1}^{\infty} u_j(z) :\ u_j \in \mathcal{U}(E_j,\, D,\, \psi)\right\} =$$

$$= \sum_{j=1}^{\infty} \sup \{u_j(z) : u_j \in \mathcal{U}(E_j, D, \psi)\} = \sum_{j=1}^{\infty} \omega(E_j, D, \psi).$$

Now we investigate the sets

$$P_j = \{z \in D : \omega(z, E_j, D, \psi) < \omega^*(z, E_j, D, \psi)\}, j \in \mathbb{N}.$$

We know that sets P_j are $m-$ polar, and their Lebesgue measure is zero. Therefore, the Lebesgue measure of $P = \bigcup_{j=1}^{\infty} P_j$ is also zero, i.e. $mes(P) = mes\left(\bigcup_{j=1}^{\infty} P_j\right) = 0.$ Now we take an upper regularization.

$$\omega^*(z, E, D, \psi) \geq \overline{\lim_{w \to z}} \sum_{j=1}^{\infty} \omega(z, E_j, D, \psi) \geq$$

$$\geq \overline{\lim_{w \to z, w \in D \setminus P}} \sum_{j=1}^{\infty} \omega^*(z, E_j, D, \psi) = \sum_{j=1}^{\infty} \omega^*(z, E_j, D, \psi).$$

The Proposition is proved. □

Proposition 4. *If $E \subset\subset D$, then* $\lim_{z \to \partial D} \omega^*(z, E, D, \psi) = 0.$

Proof. Since D is a strongly $m-$convex domain, there exists a function such that $\rho(z) \in sh_m(D) \bigcap C(D)$, $\lim_{z \to \partial D} \rho(z) = 0$. It followed that $C \cdot p(z) \in \mathcal{U}(E, D, \psi)$, where $C = \frac{\inf_{z \in E} \psi(z)}{\max_{z \in E} p(z)}$. So, from the relations $C \cdot \rho \leq \omega^*(z, E, D, \psi) \leq 0$ and $\lim_{z \to \partial D} C \cdot \rho = 0$, we get $\lim_{z \to \partial D} \omega^*(z, E, D, \psi) = 0$. The Proposition is proved. □

Proposition 5. *Let $E \subset\subset D_1$ and $D_j \subset D_{j+1}$,* $\bigcup_{j=1}^{\infty} D_j = D$, $j \in \mathbb{N}$. *Then*

$$\lim_{j \to \infty} \omega^*(z, E, D_j, \psi) = \omega^*(z, E, D, \psi).$$

Proof. According to the proposition 1, the inequality $\omega^*(z, E, D_j, \psi) \geq \omega^*(z, E, D_{j+1}, \psi)$ is valid. It follows that $\omega^*(z, E, D_j, \psi)$ is decreasing with respect to j and $\lim_{j \to \infty} \omega^*(z, E, D_j, \psi)$ exists and $\lim_{j \to \infty} \omega^*(z, E, D_j, \psi) = \omega(z) \in sh_m(D)$. Therefore the inequality

$$\lim_{j \to \infty} \omega^*(z, E, D_j, \psi) \geq \omega^*(z, E, D, \psi)$$

holds for all $z \in D$.

Now we have to show that $\lim_{j \to \infty} \omega^*(z, E, D_j, \psi) \leq \omega^*(z, E, D, \psi)$. We choose $\varepsilon_j > 0$ so that the following relationships hold $\{z \in D : \rho \leq -\varepsilon_j\} \subset D_j$, where

$\rho \in sh_m(D)$, $\lim\limits_{z\to\partial D}\rho(z) = 0$ and $\lim\limits_{j\to\infty}\varepsilon_j = 0$. Take an arbitrary $u \in \mathcal{U}(E, D_j, \psi)$ and consider the function

$$v(z) = \begin{cases} \max\{u - C\varepsilon_j,\, C\rho\},\ z \in D_j \\ C\rho,\ z \in D\backslash D_j \end{cases},$$

where $C = \frac{\inf\limits_{z\in E}\psi(z)}{\max\limits_{z\in \bar{E}} p(z)}$. It is not difficult to see that $v \in \mathcal{U}(E, D, \psi)$ and $u(z) - C\varepsilon_j \le v(z) \le \omega^*(z, E, D, \psi)$ for all $z \in D_j$. Since $u \in \mathcal{U}(E, D_j, \psi)$ is arbitrary, the inequality $\omega^*(z, E, D_j, \psi) - C\varepsilon_j \le \omega^*(z, E, D, \psi)$ holds for arbitrary $z \in D_j$. As a result, one has the inequality $\lim\limits_{j\to\infty}\omega^*(z, E, D_j, \psi) \le \omega^*(z, E, D, \psi)$ for all $z \in D$. The proof is over. □

Proposition 6. *a) let $E \subset D$ be an arbitrary set and a function $\psi(z)$ be a lower semicontinuous in $V \subset D$, where V some neighborhoods of E. Then there exists a sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$ such that*

$$\left(\lim_{j\to\infty}\omega^*(z, U_j, D, \psi)\right)^* = \omega^*(z, E, D, \psi).$$

b) let $U \subset D$ be an open set and $U = \bigcup\limits_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^0$ are a compact sets and $\psi(z)$ is an upper semicontinuous function in U. Then

$$\omega^*(z, K_j, D, \psi) \downarrow \omega^*(z, U, D, \psi).$$

Proof. a) By Lemma 2 there exists a class of countable functions $\{u_k\} \subset \mathcal{U}(E, D, \psi)$ such that $\left(\sup\limits_k u_k(z)\right)^* = \omega^*(z, E, D, \psi)$. It is evident that the functional sequence $v_j(z) = \sup\{u_1(z),\ u_2(z),\ \dots,\ u_j(z)\}$ is increasing and

$$\left(\lim_{j\to\infty} v_j(z)\right)^* = \omega^*(z, E, D, \psi).$$

Now take the sets $U_j = \left\{x \in D : v_j < \psi(z) + \frac{1}{j}\right\}$, $j \in \mathbb{N}$. The sets U_j are open because the functions $v_j(z) - \psi(z)$ are upper semicontinuous. It is easy to see, that $U_j \supset U_{j+1}$ and $v_j - \frac{1}{j} \in \mathcal{U}(U_j, D, \psi)$ for all $j \in \mathbb{N}$. By proposition 1, we get $\omega^*(z, U_j, D, \psi) \le \omega^*(z, E, D, \psi)$, $\forall j \in \mathbb{N}$. On the other hand $v_j - \frac{1}{j} \le \omega(z, U_j, D, \psi)$, $\forall j \in N$. As a result,

$$v_j - \frac{1}{j} \le \omega(z, U_j, D, \psi) \le \omega^*(z, E, D, \psi),\ \forall j \in N.$$

Now we take the limit $j \to \infty$ and the regularization:

$$\left\{\lim_{m\to\infty}\left(v_m(z) - \frac{1}{m}\right)\right\}^* \le \left\{\lim_{j\to\infty}\omega(z, U_j, D, \psi)\right\}^* \le \left\{\lim_{j\to\infty}\omega(z, E, D, \psi)\right\}^*.$$

Consequently $\left\{\lim\limits_{m\to\infty}\omega\left(z,\,U_j,\,D,\,\psi\right)\right\}^*=\omega\left(z,\,E,\,D,\,\psi\right).$ The proof of **a)** is over.

b) By proposition 1 we have $\omega^*\left(z,\,K_j,\,D,\,\psi\right)\geq\omega^*\left(z,\,K_{j+1},\,D,\,\psi\right).$ The functions $\omega^*\left(z,\,K_j,\,D,\,\psi\right)$ are decreasing $m-$subharmonic, and there are limit is $m-$subharmonic, i.e. $\lim\limits_{j\to\infty}\omega^*\left(z,\,K_j,\,D,\,\psi\right)=\omega\left(z\right),\ \omega\left(z\right)\in sh_m\left(D\right).$

By the monotony $\omega^*\left(z,\,K_j,\,D,\,\psi\right)\geq\omega^*\left(z,\,U,\,D,\,\psi\right),\ \forall j\in\mathbb{N}$ and $\omega\left(z\right)\geq\omega^*\left(z,\,U,\,D,\,\psi\right)$ for all $z\in D$. Now we have to show that $\omega\left(z\right)\leq\omega^*\left(z,\,U,\,D,\,\psi\right).$ Since the function $\psi\left(z\right)$ is an upper semicontinuous in U for $\forall z\in\bigcup\limits_{j=1}^{\infty}K_j=U$, $\exists j_0,\,\forall j>j_0,\ \omega^*(z,K_j,D,\psi)\leq\psi\left(z\right).$ It follows from this that $\left.w\left(z\right)\right|_K\leq\left.\psi\left(z\right)\right|_K.$ Therefore $w\left(z\right)\in\mathcal{U}\left(U,D,\psi\right)$ and $w\left(z\right)\leq\omega^*\left(z,U,\,D,\,\psi\right).$ The proof of the Proposition is over. □

Proposition 7. *The inequality*

$$-\inf_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,E,D\right)\leq\omega^*\left(z,E,D,\psi\right)\leq-\sup_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,E,D\right),\ \forall z\in D\quad(3)$$

holds for any set $E\subset D$.

It follows from inequality (3) that The measure $\omega^*\left(z,E,D,\psi\right)$ is either nowhere 0 or identically 0. The latter holds if and only if E is $m-$polar in D.

Proof. Take an arbitrary function $u\left(z\right)\in\mathcal{U}\left(E,D\right)$ i.e. $\left.u\left(z\right)\right|_E\leq-1,\ \left.u\left(z\right)\right|_D<0.$ Since $\left.\psi\left(z\right)\right|_E<0$, we have $-\inf\limits_{z\in E}\psi\left(z\right)>0$ and $-\inf\limits_{z\in E}\psi\left(z\right)\cdot u\left(z\right)\in sh_m\left(D\right).$ Note that

$$\left.-\inf_{z\in E}\psi\left(z\right)\cdot u\left(z\right)\right|_D<0,\quad\left.-\inf_{z\in E}\psi\left(z\right)\cdot u\left(z\right)\right|_E\leq\inf_{z\in E}\psi\left(z\right)\leq\left.\psi\right|_E.$$

Consequently $-\inf\limits_{z\in E}\psi\left(z\right)\cdot u\left(z\right)\in\mathcal{U}\left(E,D,\psi\right)$ and $-\inf\limits_{z\in E}\psi\left(z\right)\cdot u\left(z\right)\leq\omega^*\left(z,E,D,\psi\right).$ As the function u is arbitrary, we get the inequality $-\inf\limits_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,\,E,\,D\right)\leq\omega^*\left(z,\,E,\,D,\,\psi\right)$ for all $z\in D$.

Now we show that the inequality $\omega^*\left(z,\,E,\,D,\,\psi\right)\leq-\sup\limits_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,\,E,\,D\right)$ holds. If $-\sup\limits_{z\in E}\psi\left(z\right)=0$, then $-\sup\limits_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,\,E,\,D\right)=0$ and $\omega^*\left(z,\,E,\,D,\,\psi\right)\leq0.$ Now let $-\sup\limits_{z\in E}\psi\left(z\right)\neq0$. Take any function $u\left(z\right)\in\mathcal{U}\left(E,\,D,\,\psi\right)$ and consider the function $-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}$. It can be easily verify that the function $-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}$ is $m-$subharmonic in D and satisfies the following conditions:

$\left.-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}\right|_D<0$ and $\left.-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}\right|_E<-1.$

Thus $-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}\in\mathcal{U}\left(E,\,D\right)$ and $-\frac{u(z)}{\sup\limits_{z\in E}\psi(z)}\leq\omega^*\left(z,\,E,\,D\right),$ Therefore the inequality $\omega^*\left(z,\,E,\,D,\,\psi\right)\leq-\sup\limits_{z\in E}\psi\left(z\right)\cdot\omega^*\left(z,\,E,\,D\right)$ follows from the arbitrariness of the function $u\left(z\right)\in\mathcal{U}\left(E,\,D,\,\psi\right).$ □

2 (m,ψ) –regularity of compacts

Note $D\subset\mathbb{C}^n$ is strongly $m-$convex a domain and $K\subset D$ a compact.

Definition 7. *A point $z^0\in K$ is said to be globally $(m,\ \psi)-$ regular if $\omega^*(z^0,K,D,\psi)=\psi(z^0)$. It is said to be locally $(m,\ \psi)-$regular if for any neighborhood B, $z^0\in B\subset\mathbb{C}^n$, the intersection $K\cap\overline{B}$ is globally $(m,\ \psi)-$regular at the point z^0, i.e. $\omega^*\left(z^0,\ K\cap\overline{B},\ D,\ \psi\right)=\psi(z^0)$. If all points of a compact K are $(m,\ \psi)-$regular, then the compact K is called a $(m,\ \psi)-$regular compact.*

Theorem 2. *Let $\psi\in C(K)\cap sh_m(K)$. A fixed point $z^0\in K\subset\mathbb{C}^n$ is locally $(m,\ \psi)-$regular if and only if it is locally $m-$regular, $\omega^*\left(z^0,\ K\cap\overline{B},\ D\right)=-1$.*

Proof. To prove this theorem, we show that if the point $z^0\in K$ is not local $m-$regular, then it is not local $(m,\ \psi)-$regular and conversely, if point $z^0\in K$ is not local $(m,\ \psi)-$regular, then it is not local $m-$regular. Let us assume that the point $z^0\in K$ is not a local $m-$regular. i.e. there exists such a ball B that $z^0\in B\subset D$: the equality $\omega^*\left(z^0,\ K\cap\overline{B},\ D\right)=-1+\delta,\ 0<\delta<1$ is valid. According to monotonicity $\omega^*\left(z^0,\ K\cap\overline{B}_1,\ D\right)\geq-1+\delta$ for any ball B_1, where $z^0\in B_1\subset B$. Therefore by Proposition 2.7

$$\omega^*\left(z^0,\ K\cap\overline{B}_1,D,\psi\right)\geq-\inf_{z\in K\cap\overline{B}_1}\psi(z)\cdot\omega^*\left(z^0,\ K\cap\overline{B}_1,D\right)\geq-\inf_{z\in K\cap\overline{B}_1}\psi(z)(-1+\delta)$$

Since $\psi(z)$ is continuous, then choosing the neighborhood B_1 small enough we have $\min\limits_{x\in K\cap\overline{B}_1}\psi(z)\geq\frac{\psi(z^0)}{1-\delta^2}$. From this inequality we get the relation

$$\omega^*\left(z^0,\ K\cap\overline{B}_1,D,\psi\right)\geq\inf_{z\in K\cap\overline{B}_1}\psi(z)\cdot(1-\delta)\geq\frac{\psi(z^0)}{1-\delta^2}\cdot(1-\delta)=\frac{\psi(z^0)}{1+\delta}>\psi\left(z^0\right).$$

Therefore $z^0\in K$ is not local $(m,\ \psi)-$regular. Now we have to show that if the point $z^0\in K$ is not local $(m,\ \psi)-$regular, then it is not local $m-$regular.
Suppose $z^0\in K$ is not $(m,\ \psi)-$regular i.e., there exists the ball B, $z^0\in B\subset D$, such that $\omega^*\left(z^0,\ K\cap\overline{B},D,\psi\right)=\psi(z^0)+\varepsilon,\ \ 0<\varepsilon<-\psi(z^0)$. By using the previous technique we get $\omega^*\left(z^0,\ K\cap\overline{B}_1,D,\psi\right)\geq\psi(z^0)+\varepsilon$ for any ball B_1, where $z^0\in B_1\subset B$. Therefore by Proposition 2.7

$$\psi\left(z^0\right)+\varepsilon\leq\omega^*\left(z^0,\ K\cap\overline{B}_1,D,\psi\right)\leq-\sup_{z\in K\cap\overline{B}_1}\psi(z)\cdot\omega^*\left(z^0,\ K\cap\overline{B}_1,D\right).$$

Since $\psi(z)$ is continuous, then choosing the neighborhood B_1 small enough we conculude that $\sup\limits_{z\in K\cap\overline{B}_1}\psi(z)\leq\psi(z^0)+\varepsilon$ and

$$\psi\left(z^0\right)+\varepsilon\leq\omega^*\left(z^0,\ K\cap\overline{B}_1,D,\psi\right)\leq-\sup_{z\in K\cap\overline{B}_1}\psi(z)\cdot\omega^*\left(z^0,\ K\cap\overline{B}_1,D\right)<$$

$$<-\left(\psi\left(z^0\right)+\varepsilon\right)\cdot\omega^*\left(z^0,\ K\cap\overline{B}_1,D\right).$$

From the last inequality we get $\omega^*\left(z^0,\ K\cap\overline{B}_1,D\right)>-1$. Hence $z^0\in K$ is not the $m-$ regular point. The theorem is proved. □

In the above theorem, it is important that $\psi(z)$ is $m-$subharmonic. Only in this case this theorem is valid. Below we give example, which show that if the function $\psi(z)$ is not $m-$subharmonic, then the theorem is false.

Example 1 (Importance of $m-$subharmonicity)**.** *Let* $\psi(z) = -1-|z|^2$, $D = B(0,2) \subset \mathbb{C}^n$, $K = \overline{B}(0,1) \subset \mathbb{C}^n$.

We check the functions $\omega^*\left(z, \overline{B}(0,1), B(0,2)\right)$ and $\omega^*\left(z, \overline{B}(0,1), B(0,2), -1-|z|^2\right)$ for regularity at $z = 0$.

It is obvious that $\psi(z) = -1 - |z|^2$ is continuous but not $m-$ subharmonic. According to the maximum principle and by the definition of the weighted $m-$ subharmonic measure $\omega^*\left(0, \overline{B}(0,1), B(0,2), -1-|z|^2\right) \leq -2 < \psi(0) = -1$. Thus the point $z = 0$ is not $(m,\psi)-$ regular. However, we know that $\omega^*\left(0, \overline{B}(0,1), B(0,2)\right) = -1$. Since any closed ball is $m-$ regular.

Theorem 3. *Let K be $(m,\ \psi)-$regular a compact set and $\psi(z)$ be a continuous in the compact K. Then $\omega^*(z,\ K, D, \psi) \equiv \omega(z,\ K, D, \psi) \in C\left(\overline{D}\right)$, for any $z \in D$.*

Proof. Let K be $(m,\ \psi)-$regular a compact i.e. $\left.\omega^*(z,\ K, D, \psi)\right|_K = \left.\psi\right|_K$. It is evident that
$\omega^*(z,\ K,\ D,\ \psi) \subset \mathcal{U}(K,\ D,\ \psi)$ and $\omega^*(z,\ K, D, \psi) \equiv \omega(z,\ K, D, \psi)$.

Now we prove that $\omega^*(z,\ K, D, \psi)$ is continuous in $\overline{D}$. Since D is the strongly $m-$ convex domain, there exists a function $\rho(z) \in C(V) \cap sh_m(V)$ such that the domain D can be written in the form $D = \{\rho(z) < 0\}$, where V is a domain and $V \supset \overline{D}$. Let consider the function $p_1(z) = \frac{\min_{z\in K}\psi(z)}{\max_{z\in K}\rho(z)} \cdot \rho(z)$. It is easy to check that $\rho_1(z) \in C(V) \cap sh_m(V)$ and $\rho_1(z) \in \mathcal{U}(K,\ D, \psi)$. Now let us fix sufficiently small $\varepsilon > 0$ and construct the domain $G_\varepsilon = \{z \in D : \rho_1(z) < -\varepsilon\}$. It is obvious the domain G_ε lies compactly in the domain D i.e., $G_\varepsilon \subset\subset D$. There exists a sequence of monotone functions $u_j(z) \in sh_m(G_\varepsilon) \cap C^\infty(G_\varepsilon)$ such that $u_j(z) \downarrow \omega^*(z,\ K,\ D,\ \psi)$ holds for any $z \in G_\varepsilon$. Note that $\overline{G}_{2\varepsilon} \subset G_\varepsilon$. By Hartogs lemma $\exists j_1 \in \mathbb{N}$, $\forall j > j_1$, $\forall z \in \partial\overline{G}_{2\varepsilon}:\ \ u_j(z) < \varepsilon$. It is easily seen that $\rho_1(z)+2\varepsilon = 0$ for arbitrary $z \in \partial\overline{G}_{2\varepsilon}$ according to the structure of $G_{2\varepsilon}$. Therefore $u_j(z) - 3\varepsilon < \rho_1(z)$, $\forall j > j_1$, $\forall z \in \partial\overline{G}_{2\varepsilon}$.

According to Whitney's theorem [13], there exists some continuous function $\tilde{\psi}(z)$ in D such that $\left.\tilde{\psi}(z)\right|_K = \left.\psi(z)\right|_K$. Now we consider open sets

$$U_\varepsilon = \left\{z \in D : \omega^*(z,\ K,\ D,\ \psi) < \tilde{\psi}(z) + \varepsilon\right\}.$$

It is clear that $K \subset U_\varepsilon$. We again apply Hartogs' lemma to the pair of sets U_ε and K and get $\exists j_2 \in \mathbb{N}$, $\forall j > j_2$, $\forall z \in K:\ u_j(z) < \psi(z) + 3\varepsilon$. Let us consider the function

$$v(z) = \begin{cases} \max\{u_j(z) - 3\varepsilon\rho_1(z)\}, & z \in G_{2\varepsilon} \\ \rho_1(z), z \in D\backslash G_{2\varepsilon} & \end{cases}.$$

It is obvious that $\left.v\right|_K \leq \left.\psi\right|_K$, $\left.v\right|_D < 0$ for $\forall j > j_3 = \max\{j_1, j_2\}$. It implies

$v(z) \in \mathcal{U}(K,D,\psi)$ and $v(z) \leq \omega^*(z,K,D,\psi)$. Consequently $u_j(z) - 3\varepsilon \leq \omega^*(z,K,D,\psi) \leq u_j(z), \forall j > j_3, \forall z \in G_{2\varepsilon}$. Therefore $\omega^*(z,K,D,\psi)$ is a uniform limit of the $u_j(z)$ as $j \to \infty$, $z \in G_{2\varepsilon}$. This implys that $\omega^*(z,K,D,\psi) \in C(G_{2\varepsilon})$ and $\omega^*(z,K,D,\psi) \in C(D)$. Since D is the strongly $m-$convex and K is compact. We get $\omega^*|_{\partial D} = 0$. Finally from Proposition 4 $\omega^*(z,K,D,\psi) \in C(\overline{D})$. The theorem is proved. □

Notation. In the condution of Theorem 3 the function $\omega^*(z,K,D,\psi)$ defined in D can be continue to a domain $V \supset \overline{D}$ as continuous function

$$w(z) = \begin{cases} \max\{\omega^*(z,K,D,\psi), \rho_1(z)\}, & z \in D \\ \rho_1(z), & z \in V \backslash D. \end{cases}$$

This function is continuous and $m-$subharmonic continuation of $\omega^*(z,K,D,\psi)$, i.e. $w|_D \equiv \omega^*|_D$, $w \in C(V) \cap sh_m(V)$.

References

[1] Sadullaev A., Abdullaev B. Potential theory in the class of m-subharmonic functions. Trudy Mat. Inst. Steklova, 279, pp. 166-192 (2012). (in Russian)

[2] Sadullaev A. Pluripotential theory. Applications. Palmarium Academic Publishing, 307 p. (2012). (in Russian)

[3] Sadullaev A. Plurisubharmonic functions. Sovrem. Probl. Mat. Fund. Naprav. VINITI, Vol. 8, pp. 65-113 (1985). (in Russian)

[4] Sadullaev A. Plurisubharmonic measures and capacities on complex manifolds. Uspekhi Mat. Nauk. Vol. 36, Issue 4, pp. 53-105 (1981). (in Russian)

[5] Blocki Z. Weak solutions to the complex Hessian equation. Ann. Inst. Fourier, Grenoble. Vol. 5, No. 55, pp. 1735-1756 (2005). (in French)

[6] Bedford E., Taylor B.A. A new capacity for plurisubharmonic functions. Acts Math. Vol. 149, No. 1-2, pp. 1-40 (1982).

[7] Siciak J. Extremal plurisubharmonic functions in $\mathbb{C}^n$. Annales Polonici Mathematici. Vol. 1, No. 39, pp. 175-211 (1981).

[8] Zaharjuta V.P. Extremal plurisubharmonic functions, Hilbert scales, and the isomorphism of spaces of analytic functions of several variables. I, II. Theory of functions, functional analysis and their applications. T. 19, pp. 133-157, T. 21, p. 65-83 (1974). (in Russian)

[9] Brelot M. The Basis of the Classical Potential Theory. Mir, Moscow (1964). (in Russian)

[10] Narzillaev N.Kh. About $\psi-$ regular points. Tashkent, Acta NUUz, No. 2.2, pp. 173-175 (2017). (in Russian)

[11] Alan M.A. Weighted regularity and two problems of Sadullaev on weighted regularity. Complex Analysis and its Synergies, Springer. Vol. 5, pp. 1-7 (2019).

[12] Bloom T., Levenberg N. Weighted pluripotential theory in $\mathbb{C}^n$. American Journal of Mathematics. Vol. 125:1, pp. 57-103 (2003).

[13] Whitney H. Analytic extensions of differentiable functions defined in closed sets. Trans. American Mathematical Society, pp. 63-89 (1934).