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## $\mathcal{P}_{(m,\psi,\delta)}$ -CAPACITY AND ITS PROPERTIES

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**Kuldoshev Kobiljon**  
*National University of Uzbekistan*  
**Tashkent, Uzbekistan**  
 qobiljonmath@gmail.com

### Abstract

In this paper, the concept of the  $P_m$ -capacity introduced in the class of  $m$ -subharmonic functions is generalized by defining the weighted  $P(m, \psi, \delta)$ -capacity. Several fundamental properties of this capacity are established, including monotonicity, boundedness, limit transition for sequences, and countable subadditivity. Moreover, the relationship between the  $P(m, \psi, \delta)$ -capacity and the corresponding external  $C(m, \psi, \delta)$ -capacity is investigated, and comparison inequalities between them are derived. The obtained results extend previously known properties of the unweighted  $P_m$ -capacity to the weighted case and contribute to the development of potential theory in the class of  $m$ -subharmonic functions in complex spaces.

**Keywords:**  $m$ -subharmonic function;  $\mathcal{P}_m$ -capacity; weighted  $\mathcal{P}_m$ -capacity;  $m$ -polar set.

**MSC 2020:** 32U05, 32U20

### 1. Introduction and Preliminaries

In the complex space  $\mathbb{C}^n$ , several capacities defined in terms of plurisubharmonic functions (*psh*) and their properties have been studied (see, for example, [1], [2], [3], [4]). Later, the class of  $m$ -subharmonic functions was studied and the capacities defined by plurisubharmonic functions were generalized to the class of  $m$ -subharmonic functions (see, for example, [5], [6], [7], [8], [9], [10], [11]). In this paper, we generalize the concept of  $\mathcal{P}_m$ -capacity introduced in the class of  $m$ -subharmonic functions in [10].

Below, we recall some definitions and concepts that will be used in this paper. Let  $D$  be a bounded domain in  $\mathbb{C}^n$ .

**Definition 1.1.** A twice differentiable function  $u(z) \in C^2(D)$  is called  *$m$ -subharmonic* if, for all  $z^0 \in D$ , the following inequalities hold

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad \text{for all } k = 1, 2, \dots, n - m + 1,$$

where  $d = \partial + \bar{\partial}$ ,  $d^c = \frac{\partial - \bar{\partial}}{4i}$ , and  $\beta = dd^c |z|^2 = \frac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ .

Z. Blocki proved that if  $u, v_1, \dots, v_{n-m}$  are twice differentiable  $m$ -subharmonic functions, then the following inequality

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0$$

holds. Conversely, if a twice differentiable function  $u$  satisfies this condition for all twice differentiable  $m$ -subharmonic functions  $v_1, \dots, v_{n-m}$ , then  $u$  is necessarily  $m$ -subharmonic (see [6]). Based on this fact,  $m$ -subharmonic functions can also be defined within the class of locally integrable functions (see [10]).

**Definition 1.2.** A function  $u \in L^1_{\text{loc}}(D)$  is called  *$m$ -subharmonic* in the domain  $D \subset \mathbb{C}^n$ , if it is upper semicontinuous and for any twice differentiable  $m$ -subharmonic functions  $v_1, v_2, \dots, v_{n-m}$ , the current

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$$

is positive, i.e., for any positive test function  $\omega$  in  $D$ , we have

$$\int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0.$$

The class of  $m$ -subharmonic functions is denoted by  $sh_m(D)$ . It is clear that

$$psh = sh_1 \subset sh_2 \subset sh_m \subset \dots \subset sh_n = sh,$$

where  $sh(D)$  denotes the class of subharmonic functions in the domain  $D$ . A set  $E \subset D$  is called  $m$ -polar in  $D \subset \mathbb{C}^n$  if there exists a function

$$u \in sh_m(D), u \not\equiv -\infty, \text{ such that } u|_E = -\infty.$$

A domain  $D \subset \mathbb{C}^n$  is called  $m$ -regular if there exists an  $m$ -subharmonic function  $\rho \in sh_m(D)$  such that

$$\rho|_D < 0 \text{ and } \lim_{z \rightarrow \partial D} \rho(z) = 0.$$

Let  $\delta \in \mathbb{R}$  be a fixed constant and  $D \subset \mathbb{C}^n$  be an  $m$ -regular domain,  $E \subset D$  be any fixed set and  $\psi(z)$  be a bounded function on  $E$  such that  $\sup_{z \in E} \psi(z) < \delta$ . We denote by  $\mathcal{U}(E, D, \psi, \delta)$  the class of all functions

$u \in sh_m(D)$  such that

$$u|_E \leq \psi|_E \text{ and } u|_D < \delta.$$

We define the function

$$\omega_m(z, E, D, \psi, \delta) = \sup\{u(z) : u \in \mathcal{U}(E, D, \psi, \delta)\}.$$

**Definition 1.3.** The upper regularization

$$\omega_m^*(z, E, D, \psi, \delta) = \overline{\lim}_{w \rightarrow z} \omega_m(w, E, D, \psi, \delta)$$

is called the  $(m, \psi, \delta)$ -subharmonic measure ( $\mathcal{P}_{(m,\psi,\delta)}$ -measure) of the set  $E$  with respect to  $D$  (see [12]).

Note that  $\omega^*(z, E, D, -1, 0)$ ,  $\psi \equiv -1$ ,  $\delta = 0$ , coincides with the  $m$ -subharmonic measure of the potential theory in the class of functions  $u \in sh_m(D)$ , i.e.,

$$\omega_m^*(z, E, D, -1, 0) = \omega_m^*(z, E, D),$$

where  $\omega_m(z, E, D) = \sup\{u(z) : u \in sh_m(D), u|_K \leq -1, u|_D < 0\}$ . By Definition 1.3, the function  $\omega_m^*(z, E, D, \psi, \delta)$  is  $m$ -subharmonic in  $D$  and the inequality

$$\omega_m^*(z, E, D, \psi, \delta) \leq \delta$$

holds for all  $z \in D$ . According to Theorem 5.3 in [10], for any set  $E \subset D$ ,

$$\{z \in D : \omega_m(z, E, D, \psi, \delta) < \omega_m^*(z, E, D, \psi, \delta)\}$$

is  $m$ -polar in  $D$ .

Let the function  $\psi(z)$  be extended to the domain  $D$ , as a function from the class  $\mathcal{U}(E, D, \psi, \delta)$ , i.e., if there is a function  $\tilde{\psi} \in sh_m(D)$  such that

$$\tilde{\psi}|_E = \psi|_E \text{ and } \tilde{\psi}|_D < \delta$$

then it is obvious that

$$\omega_m(z, E, D, \psi, \delta) \geq \tilde{\psi}(z), \quad \forall z \in D,$$

and

$$\omega_m(z, E, D, \psi, \delta) = \psi(z), \quad \forall z \in E. \tag{1}$$

However, in general, the equality (1) does not hold (see [12]). In this paper, we consider a special case, in which the condition (1) is satisfied in the definition of  $\omega_m(z, E, D, \psi, \delta)$ .

Let  $K$  be a compact in the domain  $D$ .

**Definition 1.4.** A point  $z^0 \in K$  is said to be  $(m, \psi)$ -regular if  $\omega_m^*(z^0, K, D, \psi) = \psi(z^0)$ . If all points of the compact set  $K$  are  $(m, \psi)$ -regular, then  $K$  is called a  $(m, \psi)$ -regular compact (see [12]).

## 2. $\mathcal{P}_{(m,\psi,\delta)}$ -capacity

We now introduce the concept of the weighted  $\mathcal{P}_m$ -capacity using the weighted  $(m, \psi, \delta)$ -subharmonic measure. Let  $E \subset D$  be a set and  $\psi(z)$  a bounded on  $E$ . Note that, as mentioned above, we consider the case  $\sup_{z \in E} \psi(z) < \delta$ ,  $\delta \in \mathbb{R}$ , when constructing the function  $\omega^*(z, E, D, \psi, \delta)$ .

**Definition 2.1.** The quantity of the following integral

$$\mathcal{P}_m(E, D, \psi, \delta) = \int_D (\delta - \omega_m^*(z, E, D, \psi, \delta)) dV$$

is called the  $\mathcal{P}_{(m,\psi,\delta)}$ -capacity of the set  $E$  with respect to  $D$ , where  $dV$  denotes the standard Lebesgue volume measure in  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ .

Note that  $\mathcal{P}_m(z, E, D, -1, 0)$ ,  $\psi \equiv -1$ ,  $\delta = 0$ , coincides with the  $\mathcal{P}_m$ -capacity, i.e.,

$$\mathcal{P}_m(E, D, -1, 0) = \mathcal{P}_m(E, D),$$

where  $\mathcal{P}_m(E, D) = -\int_D \omega_m^*(z, E, D)dV$ . It follows from Definition 2.1 that

$$0 \leq \mathcal{P}_m(E, D, \psi, \delta) \leq \left( \delta - \inf_{z \in E} \psi(z) \right) \text{vol}(D),$$

where  $\text{vol}(D)$  denotes the volume of the domain  $D$ . Since the following set

$$\{z \in D : \omega_m(z, E, D, \psi, \delta) < \omega_m^*(z, E, D, \psi, \delta)\}$$

is  $m$ -polar (see [10]), we have

$$\mathcal{P}_m(E, D, \psi, \delta) = \int_D (\delta - \omega_m(z, E, D, \psi, \delta)) dV.$$

Moreover, since  $\omega_m(z, E, D, \psi, \delta) \equiv \delta$  if and only if  $E$  is an  $m$ -polar set (see [12]), we conclude that

$$\mathcal{P}_m(E, D, \psi, \delta) = 0,$$

if and only if,  $E$  is an  $m$ -polar set.

**Proposition 2.1.** *The following monotonicity properties hold:*

- if  $E_1 \subset E_2 \subset D$ , then  $\mathcal{P}_m(E_1, D, \psi, \delta) \leq \mathcal{P}_m(E_2, D, \psi, \delta)$ ;
- if  $\psi_1 \leq \psi_2$ , then  $\mathcal{P}_m(E, D, \psi_1, \delta) \geq \mathcal{P}_m(E, D, \psi_2, \delta)$ ;
- if  $E \subset D_1 \subset D_2$ , then  $\mathcal{P}_m(E, D_1, \psi, \delta) \leq \mathcal{P}_m(E, D_2, \psi, \delta)$ ;
- if  $\sup_{z \in E} \psi(z) < \delta_1 < \delta_2$ , then  $\mathcal{P}_m(E, D, \psi, \delta_1) \leq \mathcal{P}_m(E, D, \psi, \delta_2)$ .

**Proof.** Since the proof of the initial three inequalities stated above follows easily from the definition of the  $\mathcal{P}_{(m,\psi,\delta)}$ -capacity, we will prove only the last one. Take an arbitrary function  $u(z)$  from the family  $\mathcal{U}(E, D, \psi, \delta_2)$ . Clearly, the inequalities

$$(u - \delta_2 + \delta_1)|_E \leq \psi \quad \text{and} \quad (u - \delta_2 + \delta_1)|_D < \delta_1$$

hold, which implies that

$$u(z) - \delta_2 + \delta_1 \leq \omega_m^*(z, E, D, \psi, \delta_1).$$

Since  $u(z)$  is arbitrary in the family  $\mathcal{U}(E, D, \psi, \delta_2)$ , it follows that

$$\delta_1 - \omega_m^*(z, E, D, \psi, \delta_1) \leq \delta_2 - \omega_m^*(z, E, D, \psi, \delta_2)$$

and therefore we obtain

$$\mathcal{P}_m(E, D, \psi, \delta_1) \leq \mathcal{P}_m(E, D, \psi, \delta_2).$$

The proof is complete. □

Below, we state the theorem that expresses our main result.

**Theorem 2.1.** *We have the following properties.*

1. If  $\{K_j\}$  is a decreasing sequence of compact sets in  $D$  and  $K = \bigcap K_j$  and assume that  $\psi(z)$  is lower semicontinuous in a neighborhood  $V \supset K$  of the set  $K$  with  $V \subset D$ , then

$$\mathcal{P}_m(K, D, \psi, \delta) = \lim_{j \rightarrow \infty} \mathcal{P}_m(K_j, D, \psi, \delta);$$

2. If  $\{E_j\}$  is an increasing sequence of arbitrary subsets of  $D$  and  $E = \bigcup E_j$ , then

$$\mathcal{P}_m(E, D, \psi, \delta) = \lim_{j \rightarrow \infty} \mathcal{P}_m(E_j, D, \psi, \delta);$$

3. The  $\mathcal{P}_{(m,\psi,\delta)}$ -capacity is countably subadditive, that is, if  $E_j \subset D$  for every  $j \in \mathbb{N}$ , then

$$\mathcal{P}_m \left( \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) \leq \sum_{j=1}^{\infty} \mathcal{P}_m(E_j, D, \psi, \delta).$$

*Proof.* First, we present the proof of part (1). Fix an arbitrary  $\varepsilon > 0$  and an arbitrary point  $z^0 \in D$ . Then there exists a function  $u \in sh_m(D)$  such that

$$u|_K \leq \psi|_K, \quad u|_D < \delta, \quad \text{and} \quad u(z^0) > \omega_m(z^0, K, D, \psi, \delta) - \varepsilon.$$

Then, since  $\psi(z)$  is lower semicontinuous in  $V$ , the set  $U = \{z \in V : u(z) < \psi(z) + \varepsilon\}$  is open and satisfies  $K \subset U$ . Hence, there exists  $j_0 \in \mathbb{N}$  such that  $K_j \subset U$  for all  $j > j_0$ . Thus, for all  $j > j_0$ ,

$$u(z) - \varepsilon \leq \omega_m(z, K_j, D, \psi, \delta) \leq \omega_m(z, K, D, \psi, \delta).$$

Therefore,

$$\omega_m(z^0, K, D, \psi, \delta) - 2\varepsilon \leq \omega_m(z^0, K_j, D, \psi, \delta) \leq \omega_m(z^0, K, D, \psi, \delta).$$

Since the point  $z^0$  was chosen arbitrarily in  $D$ , we conclude that for all  $z \in D$ ,

$$\omega_m(z, K, D, \psi, \delta) = \lim_{j \rightarrow \infty} \omega_m(z, K_j, D, \psi, \delta).$$

By Lebesgue's theorem (see, for example, [13]), we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_m(K_j, D, \psi, \delta) &= \lim_{j \rightarrow \infty} \int_D (\delta - \omega_m(z, K_j, D, \psi, \delta)) dV \\ &= \int_D \lim_{j \rightarrow \infty} (\delta - \omega_m(z, K_j, D, \psi, \delta)) dV = \int_D (\delta - \omega_m(z, K, D, \psi, \delta)) dV = \mathcal{P}_m(K, D, \psi, \delta). \end{aligned}$$

The proof of part (1) is complete.

Now we prove part 2.

For any  $j \in \mathbb{N}$ , since  $E_j \subset E_{j+1}$ , the sequence of functions  $\omega_m^*(z, E_j, D, \psi, \delta)$  is decreasing with respect to  $j$ . As a result, the function

$$v(z) = \lim_{j \rightarrow \infty} \omega_m^*(z, E_j, D, \psi, \delta)$$

is  $m$ -subharmonic in  $D$  and  $\omega_m^*(z, E, D, \psi, \delta) \leq v(z)$  for all  $z \in D$ . Since for all  $j \in \mathbb{N}$ , the following sets

$$\{z \in D : \omega_m(z, E_j, D, \psi, \delta) < \omega_m^*(z, E_j, D, \psi, \delta)\}$$

are  $m$ -polar in  $D$ , the set

$$F = \{z \in E : v(z) > \psi(z)\}$$

is also  $m$ -polar in  $D$ . Then there exists a nonpositive  $m$ -subharmonic function  $u$  in  $D$  such that  $u|_E = -\infty$  and  $u|_D \not\equiv -\infty$ . It is easy to verify that for any  $\varepsilon > 0$ , the function  $v(z) + \varepsilon u(z)$  belongs to the class  $\mathcal{U}(E, D, \psi, \delta)$  and satisfies

$$v(z) + \varepsilon u(z) \leq \omega_m(z, E, D, \psi, \delta).$$

Hence,

$$v(z) \leq \omega_m^*(z, E, D, \psi, \delta).$$

Thus, we have

$$\omega_m^*(z, E, D, \psi, \delta) = \lim_{j \rightarrow \infty} \omega_m^*(z, E_j, D, \psi, \delta). \quad (2)$$

By Levi's theorem (see, for example, [13]) and according to (2),

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_m(E_j, D, \psi, \delta) &= \lim_{j \rightarrow \infty} \int_D (\delta - \omega_m^*(z, E_j, D, \psi, \delta)) dV \\ &= \int_D \lim_{j \rightarrow \infty} (\delta - \omega_m^*(z, E_j, D, \psi, \delta)) dV = \int_D (\delta - \omega_m^*(z, E, D, \psi, \delta)) dV = \mathcal{P}_m(E, D, \psi, \delta). \end{aligned}$$

The proof of part 2 is complete.

Finally, we present the proof of part 3.

Let us choose an arbitrary function  $u_j(z)$  from each family  $\mathcal{U}_m(E_j, D, \psi, \delta)$  and consider the function defined by

$$\delta + \sum_{j=1}^{\infty} (u_j(z) - \delta).$$

Since the limit of a decreasing sequence of  $m$ -subharmonic functions is again  $m$ -subharmonic, the function defined above is  $m$ -subharmonic in the domain  $D$  and it belongs to the family  $\mathcal{U}_m(\cup_{j=1}^{\infty} E_j, D, \psi, \delta)$ . Thus, for any  $z \in D$ ,

$$\sum_{j=1}^{\infty} (u_j(z) - \delta) \leq \omega_m \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) - \delta.$$

Since each function  $u_j(z)$  is an arbitrary element of the family  $\mathcal{U}_m(E_j, D, \psi, \delta)$ , it follows that for any  $z \in D$ ,

$$\sum_{j=1}^{\infty} (\omega_m(z, E_j, D, \psi, \delta) - \delta) \leq \omega_m \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) - \delta. \quad (3)$$

It is known that the sets

$$\{z \in D : \omega_m(z, E_j, D, \psi, \delta) < \omega_m^*(z, E_j, D, \psi, \delta)\}$$

and

$$\left\{ z \in D : \omega_m \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) < \omega_m^* \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) \right\}$$

are  $m$ -polar. Therefore, by inequality (3) and according to Fatou's theorem (see, for example, [13]), we have

$$\begin{aligned} \mathcal{P}_m \left( \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) &= \int_D \left( \delta - \omega_m^* \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) \right) dV = \int_D \left( \delta - \omega_m \left( z, \bigcup_{j=1}^{\infty} E_j, D, \psi, \delta \right) \right) dV \\ &\leq \int_D \left( \sum_{j=1}^{\infty} (\delta - \omega_m(z, E_j, D, \psi, \delta)) \right) dV \leq \sum_{j=1}^{\infty} \int_D (\delta - \omega_m(z, E_j, D, \psi, \delta)) dV \\ &= \sum_{j=1}^{\infty} \int_D (\delta - \omega_m^*(z, E_j, D, \psi, \delta)) dV = \sum_{j=1}^{\infty} \mathcal{P}_m(E_j, D, \psi, \delta). \end{aligned}$$

The proof is complete.  $\square$

**Proposition 2.2.** *Let  $E \subset D$  be an arbitrary set and a function  $\psi(z)$  be lower semicontinuous in  $V \subset D$ , where  $V$  is some neighborhood of  $E$ . Then for any  $\varepsilon > 0$ , there exists an open set  $U \supset E$  such that*

$$\mathcal{P}_m(U, D, \psi, \delta) - \mathcal{P}_m(E, D, \psi, \delta) < \varepsilon.$$

**Proof.** If the function  $\psi(z)$  is lower semicontinuous in some neighborhood of the set  $E$ , then there exists a sequence of open sets  $U_j \supset E$ ,  $U_j \supset U_{j+1}$  such that

$$\left( \lim_{j \rightarrow \infty} \omega_m^*(z, U_j, D, \psi, \delta) \right)^* = \omega_m^*(z, E, D, \psi, \delta), \quad (4)$$

which was proved in [12]. Thus, according to Lebesgue's theorem (see, for example, [13]) and by (4), we obtain

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_m(U_j, D, \psi, \delta) &= \lim_{j \rightarrow \infty} \int_D (\delta - \omega_m(z, U_j, D, \psi, \delta)) dV \\ &= \int_D \lim_{j \rightarrow \infty} (\delta - \omega_m(z, U_j, D, \psi, \delta)) dV = \int_D (\delta - (\lim_{j \rightarrow \infty} \omega_m(z, U_j, D, \psi, \delta))^* dV \\ &= \int_D (\delta - \omega_m^*(z, E, D, \psi, \delta)) dV = \mathcal{P}_m(E, D, \psi, \delta). \end{aligned}$$

Consequently, for any  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that for all  $j > j_0$ , we have

$$\mathcal{P}_m(U_j, D, \psi, \delta) - \mathcal{P}_m(E, D, \psi, \delta) < \varepsilon.$$

The proof is complete. □

**Definition 2.2.** The following quantity

$$C_m(K, D, \psi, \delta) = \inf \left\{ \int_D (dd^c u)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m(D) \cap C(D), u|_K \leq \psi|_K, \lim_{z \rightarrow \partial D} u(z) \geq \delta \right\}$$

is called the  $(m, \psi, \delta)$ -capacity of  $K$  with respect to  $D$ .

In [14], it was shown that for  $\psi \in C(E)$ ,  $C_m^*(E, D, \psi, \delta) = 0$  if and only if  $E$  is  $m$ -polar. Consequently,  $C_m^*(E, D, \psi, \delta) = 0$  if and only if  $\mathcal{P}_m(E, D, \psi, \delta) = 0$ , where  $C_m^*(E, D, \psi, \delta)$  denotes the  $(m, \psi, \delta)$ -external capacity of the set  $E$  with respect to  $D$ , defined by

$$C_m^*(E, D, \psi, \delta) = \inf \{ C_m(U, D, \psi, \delta) : E \subset U, U \subset D, U \text{ is open} \},$$

$$C_m(U, D, \psi, \delta) = \sup \{ C_m(K, D, \psi, \delta) : K \subset U, K \text{ is compact} \}.$$

We recall an important theorem from [3], which will be needed to estimate the  $(m, \psi, \delta)$ -external capacity in terms of the  $\mathcal{P}_{(m,\psi,\delta)}$ -capacity.

**Theorem 2.2.** Let  $u(z) \in sh_m(B) \cap C(B)$ , where  $B = \{z \in \mathbb{C}^n : |z| < 1\}$ . Then for any  $0 < r < 1$  and  $1 \leq k \leq n - m + 1$ , the following inequality holds:

$$\int_0^r dt \int_{|z|^2 \leq t} (dd^c u)^k \wedge \beta^{n-k} \leq (C - c) \int_{|z|^2 \leq r} (dd^c u)^{k-1} \wedge \beta^{n-k+1},$$

where  $C = \sup_{z \in B} u(z)$  and  $c = \inf_{z \in B} u(z)$ .

Let  $B = B(0, 1)$  denote the unit ball centered at the origin.

**Proposition 2.3.** Suppose that  $E \subset B(0, r) \subset\subset B(0, 1)$ , with  $0 < r < 1$  and  $\psi \in C(E)$ . Then the following inequalities hold.

1.  $C_m^*(E, B, \psi, \delta) \leq \frac{(\delta - \inf_E \psi)^{n-m} (n-m+1)! \mathcal{P}_m(E, B, \psi, \delta)}{(1-r^2)^{n-m+1}}$ .
2.  $(\mathcal{P}_m(E, B, \psi, \delta))^{\frac{1}{n-m+1}} \leq C(r, \psi, \delta) \cdot C_m^*(E, B, \psi, \delta)$ , where  $C(r, \psi, \delta)$  is a constant depending on  $r, \psi$ , and  $\delta$ .

**Proof.** It suffices to prove the inequalities for the case when the set  $E$  is a  $(m, \psi, \delta)$ -regular compact set. First, we prove the first inequality. If  $E$  is an  $(m, \psi, \delta)$ -regular compact set, then by Theorem 2.1 in [12], the function  $\omega^*(z, E, B, \psi, \delta)$  is continuous in  $B$ . By applying Theorem 2.2.  $k = n - m + 1$  times, we obtain the following inequality:

$$\begin{aligned} & \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-m}} dt_{n-m+1} \int_{|z|^2 \leq t_{n-m+1}} (dd^c \omega^*(z, E, B, \psi, \delta))^{n-m+1} \wedge \beta^{m-1} \\ & \leq \left( \delta - \inf_{z \in E} \psi(z) \right)^{n-m} \int_B (\delta - \omega^*(z, E, B, \psi, \delta)) \beta^n = \left( \delta - \inf_E \psi \right)^{n-m} \cdot \mathcal{P}_m(E, B, \psi, \delta). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-m}} dt_{n-m+1} \int_{|z|^2 \leq t_{n-m+1}} (dd^c \omega^*(z, E, B, \psi, \delta))^{n-m+1} \wedge \beta^{m-1} \\ & \geq \int_{r^2}^1 dt_1 \int_{r^2}^{t_1} dt_2 \cdots \int_{r^2}^{t_{n-m}} dt_{n-m+1} \int_{|z|^2 \leq r^2} (dd^c \omega^*(z, E, B, \psi, \delta))^{n-m+1} \wedge \beta^{m-1} \\ & = \frac{(1-r^2)^{n-m+1}}{(n-m+1)!} \int_{B(0,r)} (dd^c \omega^*(z, E, B, \psi, \delta))^{n-m+1} \wedge \beta^{m-1} = \frac{(1-r^2)^{n-m+1}}{(n-m+1)!} C_m(E, B, \psi, \delta). \end{aligned}$$

Consequently, for any  $E \subset B(0, r)$ ,  $r < 1$ , we have

$$C_m^*(E, B, \psi, \delta) \leq \frac{\left(\delta - \inf_E \psi\right)^{n-m} (n-m+1)! \mathcal{P}_m(E, B, \psi, \delta)}{(1-r^2)^{n-m+1}}.$$

We now proceed to the proof of the second inequality. In [10], in the unweighted case, i.e., when  $\psi \equiv -1$  and  $\delta = 0$ , it was proved that there exists a constant  $M(r) > 0$  such that for any  $E \subset B(0, r)$ ,

$$\left(\mathcal{P}_m(E, B)\right)^{\frac{1}{n-m+1}} \leq M(r) \cdot C_m^*(E, B). \quad (5)$$

According to Proposition 1.3 in our previous work [12], we obtain the inequality

$$\mathcal{P}_m(E, B, \psi, \delta) \leq \left(\delta - \inf_E \psi\right) \mathcal{P}_m(E, B). \quad (6)$$

From the relation obtained on page 11 of [14],

$$\left[\delta - \sup_{z \in E} \psi(z)\right]^{n-m+1} C_m^*(E, D) \leq C_m^*(E, D, \psi, \delta),$$

together with (5) and (6), we obtain the following inequality

$$\left(\mathcal{P}_m(E, B, \psi, \delta)\right)^{\frac{1}{n-m+1}} \leq C(r, \psi, \delta) \cdot C_m^*(E, B, \psi, \delta).$$

The proof is complete. □

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**Received: 28/07/2025**

**Accepted: 31/08/2025**

**Cite this article**

Kuldoshev K.  $\mathcal{P}_{(m,\psi,\delta)}$ -capacity and its properties. *Bull. Inst. Math.*, 2025, Vol.8, No 5, pp. 40-47