Uzbek Mathematical Journal
2025, Volume 69, Issue 2, pp.153-164
DOI: 10.29229/uzmj.2025-2-14

Weighted (m,ψ)-capacity $C_m(K,D,\psi)$ of a condenser (K,D)

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Dedicated to the 80 th birthday of Academician Shavkat Arifdzhanovich Alimov and the 70 th birthday of Professor Ravshan Radjabovich Ashurov

Abstract. In this paper, we introduce the concept of the capacity of a condenser (K,D), with a weighted function $\psi(z)\in C(K)$ for a compact set $K\subset D$, where $D\subset\mathbb{C}^n$ is a domain, in the class of m- subharmonic functions.

Keywords: m-subharmonic function, m-subharmonic measure, weighted (m,ψ)-subharmonic measure, weighted (m,ψ)- capacity.

MSC (2020): 32U05, 32U20.

1. Introduction

Capacity of condensers is one of the important concepts of the potential theory and it has been extensively studied by many researchers. In the works of A. Sadullaev and B. Abdullaev [7], [1], Z. Blocki [2], S. Dinev, S. Kolodziej [3] the concept of condenser capacity were introduced, in the class of m-subharmonic functions, where extensive research was conducted, leading to significant results.

Recall that m- subharmonic functions in the domain $D\subset\mathbb{C}^n$ is defined using the operators

$$(dd^cu)^k\wedge\beta^{n-k},\qquad 1\le k\le n, \tag{1.1}$$

where $d=\partial+\bar\partial$, $d^c=\frac{\partial-\bar\partial}{4i}$ and $\beta=dd^c|z|^2=\frac{i}{2}\sum\limits_{i=1}^{n}dz_i\wedge d\bar z_i$ is standard canonical (1,1) form in $\mathbb{C}^n$. Then $\frac{1}{n!}\beta^n=dV_n$ is volume form in $\mathbb{C}^n$. Operator (1.1) gives the Laplace operator for $k=1$ and the Monge-Ampere operator for $k=n$. The operator (1.1) is called the complex Hessians operator, as it can be shown for $u\in C^2(D)$ that

$$(dd^cu)^k\wedge\beta^{n-k}=k!(n-k)!H_k(u)\beta^n,$$

where

$$H_k(u)=\sum_{1\le j_1<j_2<\ldots<j_k\le n}\lambda_{j_1}\cdot\lambda_{j_2}\cdot\ldots\cdot\lambda_{j_k}$$

is the Hessian of dimension k of the eigenvalue vector $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$ of the matrix

$$(u_{j,\bar l}),\ \ u_{j,\bar l}=\frac{\partial^2u}{\partial z_j\partial\bar z_l},\ \ j,l=1,2,...,n.$$

Below, we outline the definition of subharmonic function and highlight its key concepts.

Definition 1.1. Twice smooth function $u(z)\in C^2(D)$ is called m-subharmonic, $u(z)\in sh_m(D)$, if the conditions

$$(dd^cu)^k\wedge\beta^{n-k}\ge 0\,,\ \forall k=1,\,2,...,\,n-m+1$$

holds at each point $z\in D$.

Z. Blocki proved that for all twice differentiable m-subharmonic functions $u,\ v_1,\ v_2,\ ...,\ v_{n-m}$ it is true

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{1.2}$$

Moreover, if a twice differentiable function u satisfies (1.2) for all twice differentiable m-subharmonic functions $v_1,\ v_2,\ \dots,\ v_{n-m}$ then u is necessarily m-subharmonic function [2]. Using this, we can define m-subharmonic functions in the class of the upper semicontinuous functions.

Definition 1.2. A function $u(z) \in L^1_{loc}(D)$ is called m-subharmonic in the domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice differentiable m-subharmonic functions $v_1,\ v_2,\ \dots,\ v_{n-m}$ the current

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}$$

defined as

$$[dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1}](\omega) =$$
$$= \int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0}(D)$$

is positive, i.e.

$$\int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge ... \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}(D),\ \omega \geq 0.$$

Class of m-subharmonic functions we denote as $sh_m(D)$. It is clear, that

$$psh = sh_1 \subset sh_2 \subset sh_m \subset ... \subset sh_n = sh. \tag{1.3}$$

Definition 1.3. A set $E \subset D$ is called m-polar in $D \subset \mathbb{C}^n$ if there exist a function $u(z) \in sh_m(D)$, $u(z) \not\equiv -\infty$, such that $u|_E = -\infty$.

(1.3) follows , that a m-polar set is polar in the sense of the classical potential theory, so that for m-polar set $E \subset D$ the Hausdorf measure $H_{2n-2+0}(E) = 0$.

Definition 1.4. A domain $D \subset \mathbb{C}^n$ is called m-regular if there exists a m-subharmonic function $\rho \in sh_m(D)$ such that $\rho|_D < 0$, $\lim\limits_{z \to \partial D} \rho(z) = 0$, i.e. $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. It is called strongly m-regular if the function $\rho \in sh_m(D^+) \bigcap C^2(D^+)$ and is strongly m-subharmonic in D^+, where D^+ is a neighborhood of the closure $\bar{D}$.

The well-known comparison principle is one of the important inequalities between m-subharmonic functions and the integrals of their Hessians [7]. Since we have made extensive use of the comparison principle in our work, we present it.

Theorem 1.5. *If $u, v \in sh_m(D) \bigcap L^\infty_{loc}(D)$ and the set $F = \{z \in D : u(z) < v(z)\} \subset\subset D$, then*

$$\int\limits_F (dd^c u)^k \wedge \beta^{n-k} \geq \int\limits_F (dd^c v)^k \wedge \beta^{n-k},\ 1 \leq k \leq n-m+1. \tag{1.4}$$

2. m-subharmonic measure and the capacity of a condenser (K, D).

The m-subharmonic measure is defined as an extremal function in the class of m-subharmonic (sh_m) functions. Let $E \subset D$ be a subset of the domain $D \subset \mathbb{C}^n$. For the sake of simplicity, we assume that D is a bounded and strongly m-regular domain, We denote by $\mathcal{U}(E, D)$ the class of all functions $u \in sh_m(D)$, such that $u|_E \leq -1$, $u|_D < 0$ and we define

$$\omega(z, E, D) = \sup\{u(z) :\ u(z) \in \mathcal{U}(E, D)\}.$$

Definition 2.1. The regularization $\omega^*(z,E,D)=\overline{\lim_{w\to z}}\,\omega(w,\,E,D)$ is called the m-subharmonic measure ($\mathcal{P}_m-$ measure) of E with respect to D (see [7], [1]).

Let $D\subset\mathbb{C}^n$ be a domain and $K\subset D$ a compact.

Definition 2.2. A point $z^0\in K$ is said to be globally m-regular if $\omega^*(z^0,K,D)=-1$. It is said to be locally m-regular if for any neighborhood $B\neq\varnothing$, $z^0\in B\subset\mathbb{C}^n$, the intersection $K\bigcap\bar{B}$ is globally m-regular at the point z^0, i.e. $\omega^*(z^0,K\bigcap\overline{B},D)=-1$. If all points of a compact set K are globally (or locally) m-regular, then the compact set K is called to be globally (or locally) m-regular compact.

m-capacity of the condenser (K,D) is introduced as follows, using the operator

$$(dd^cu)^{n-m+1}\wedge\beta^{m-1}.$$

Definition 2.3. Let K be a compact subset of $D\subset\mathbb{C}^n$. The following quantity

$$C_m(K,D)=\inf\left\{\int\limits_D(dd^cu)^{n-m+1}\wedge\beta^{m-1}:u\in sh_m(D)\bigcap C(D),\ u|_K\leq-1,\ \underline{\lim}_{z\to\partial D}u(z)\geq0\right\}$$

is called the m-capacity of the condenser (K,D).

Note that, for m-regular compact $K\subset D$ the m-capacity of the condenser is equal

$$C_m(K,D)=\int\limits_K(dd^c\omega^*(z,K,D))^{n-m+1}\wedge\beta^{m-1}=\int\limits_D(dd^c\omega^*(z,K,D))^{n-m+1}\wedge\beta^{m-1}.$$

Moreover, external capacity $C_m^*(E,D)=0$ if and only if E is m-polar set in D.

3. Weighted m-subharmonic measure.

(m,ψ)-subharmonic measure and its properties. Let $D\subset\mathbb{C}^n$ be a strongly m-regular domain, $E\subset D$ be any fixed set and $\psi(z)$ be a bounded and negative function in E. We denote by $\mathcal{U}(E,D,\psi)$ the class of all functions $u(z)\in sh_m(D)$, such that

$$u|_E\leq\psi|_E,\ u|_D<0.$$

Using this family of functions, we define

$$\omega(z,E,D,\psi)=\sup\{u(z):u(z)\in\mathcal{U}(E,D,\psi)\}.$$

Definition 3.1. The regularization

$$\omega^*\left(z,\,E,\,D,\psi\right)=\overline{\lim_{w\to z}}\,\omega\left(w,\,E,\,D,\psi\right)$$

is called weighted $(m,\psi)-$subharmonic measure ($\mathcal{P}_{(m,\psi)}-$measure) of the set E with respect to D.

Note that $\omega^*(z,E,D,-1)$ $(\psi\equiv-1)$, coincides with the m-subharmonic measure of the potential theory in the class of $u(z)\in sh_m(D)$, i.e. $\omega^*(z,E,D,-1)=\omega^*(z,E,D)$.

As one can see from the definition 3.1, the function $\omega^*(z,\,E,\,D,\psi)$ is m-subharmonic in D. If $0\leq\inf\limits_{z\in E}\psi(z)$, then $\omega^*(z,\,E,\,D,\psi)=0$, $\forall z\in D$. Therefore, in this paper, we will consider the special case, where $\sup\limits_{z\in E}\psi(z) < 0$ is satisfied. The weighted (m,ψ)-subharmonic measure satisfies the properties m-subharmonic measure and the inequality

$$-\inf_{z\in E}\psi(z)\cdot\omega^*(z,E,D) \le \omega^*(z,E,D,\psi) \le -\sup_{z\in E}\psi(z)\cdot\omega^*(z,E,D) \tag{3.1}$$

holds for any set $E\subset D$ and for all $z\in D$ (see [5]).

$(m,\ \psi)-$ **regularity of compacts.** Let the function $\psi(z)$ be extended to the domain D as a function from the class $\mathcal{U}(E,D,\psi)$ i.e. if there is a function

$$\tilde{\psi}\in sh_m(D), \tilde{\psi}|_E=\psi|_E,\ \tilde{\psi}|_D<0, \tag{3.2}$$

then it is obvious $\omega(z,E,\ D,\psi)\ge\tilde{\psi}(z)$, $\forall z\in D$ and

$$\omega(z,E,\ D,\psi)=\psi(z),\ \forall z\in E. \tag{3.3}$$

However, if condition (3.2) is not satisfied, then, in general, equality (3.3) does not hold, a priori, it may happen that $\omega(z,E,\ D,\ \psi)<\psi(z)$ at some points $z\in E$ (see [5]).

Below we assume that the condition (3.3) satisfied. We also assume, that $D\subset\mathbb{C}^n$ is a strongly m-regular domain and $K\subset D$ is a compact.

Definition 3.2. A point $z^0\in K$ is said to be globally (m,ψ)-regular if $\omega^*(z^0,\ K,D,\psi)=\psi(z^0)$. It is said to be locally (m,ψ)-regular if for any neighborhood B, $z^0\in B\subset\mathbb{C}^n$, the intersection $K\bigcap\bar{B}$ is globally (m,ψ)-regular at the point z^0, i.e. $\omega^*(z^0,\ K\bigcap\overline{B},\ D,\ \psi)=\psi(z^0)$. If all points of a compact set K are globally (or locally) $(m,\psi)-$regular, then the compact set K is called a globally (or locally) $(m,\psi)-$regular compact.

Now we present some important theorems on the (m,ψ)-regularity of a compact K.

Theorem 3.3. *Let K be (m,ψ)-regular compact set and $\psi(z)$ be continuous in the compact K. Then*

$$\omega^*(z,K,D,\psi)\equiv\omega(z,K,D,\psi)\in C(\bar{D})$$

for any $z\in D$.

Theorem 3.4. *Let $\psi\in C(K)$ and condition (3.3) be satisfied, i.e.*

$$\omega(z,K,\ D,\psi)=\psi(z),\forall z\in K.$$

A fixed point $z^0\in K\subset\mathbb{C}^n$ is locally (m,ψ)-regular if and only if it is locally m-regular, $\omega^*(z^0,\ K\bigcap\overline{B},D)=-1\ \ \forall B=B(z^0,r),\ \ r>0.$

Theorem 3.5. *Let the function $\psi(z)$ be continuous in the compact K and extended to $\mathcal{U}(K,D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+\supset\bar{D}$ of the closure $\bar{D}$, i.e, there exists a function $\tilde{\psi}$ such that it is strictly m-subharmonic in the domain D^+ and $\tilde{\psi}|_K=\psi|_K,\tilde{\psi}|_D<0$. Then a fixed point $z^0\in K\subset D$ is locally (m,ψ)-regular if and only if it is globally (m,ψ)-regular.*

Theorem 3.3, Theorem 3.4 and Theorem 3.5 mentioned above were proven in our previous works (see [5], [6]).

From the Theorem 3.4 and the Theorem 3.5, we obtain several important corollaries.

Corollary 3.6. *If the compact set $K\subset D$ is globally (m,ψ)- regular, where the function $\psi(z)$ is extended to $\mathcal{U}(K,\ D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+\supset\bar{D}$ of closure $\bar{D}$, then K is locally m-regular.*

Corollary 3.7. *If ψ_1 and ψ_2 are continuous in the compact K and extended to $\mathcal{U}(K,D,\psi_1)$ and $\mathcal{U}(K,D,\psi_2)$ as strictly m-subharmonic functions in some neighbourhood $D^+ \supset \bar{D}$ of closure $\bar{D}$, respectively, then the point $z^0 \in K \subset D$ is $(m,\psi_1)-$regular if and only if it is $(m,\psi_2)-$regular.*

Corollary 3.8. *If the compact set $K \subset D$ is globally (m,ψ)-regular, where $\psi(z)$ is continuous in the compact K and extended to $\mathcal{U}(K,D,\psi)$ as a strictly m-subharmonic function in some neighbourhood $D^+ \supset \bar{D}$ of closure $\bar{D}$, then K is not m-polar at each of its point. It means that for any $z^0 \in K$ and for any neighborhood $B \subset D$, $z^0 \in B$ the intersection $E = B \bigcap K$ is not m-polar.*

Maximality of (m,ψ)-subharmonic measures. Maximal functions are one of the important concepts of the potential theory and they are analog of harmonic functions in the class of m-subharmonic functions. We remember,

Definition 3.9. A function $u \in sh_m(D)$ is called maximal in the domain $D \subset \mathbb{C}^n$ if it satisfies the dominance principle within the class of m-subharmonic functions, i.e., if $\forall v \in sh_m(D)$: $\underline{\lim}_{z\to\partial D} (u(z) - v(z)) \geq 0$, then $u(z) \geq v(z)$, $\forall z \in D$ (see [7]).

Let $B(0,1) \subset \mathbb{C}^n$ be a ball and $\varphi(\xi)$ be a continuous function defined on the boundary ∂B. Construct the function

$$w(z) = \sup\{u(z) : \ u \in \mathcal{U}(\varphi, B)\}$$

using the class of $\mathcal{U}(\varphi,B) = \{u \in sh_m(D) \bigcap C(\bar{D}) : u|_{\partial B} \leq \varphi\}$. Z. Blocki in [2] proved that the function $w(z)$ is continuous and maximal, $w \in sh_m(B) \bigcap C(\bar{B})$, $w|_{\partial B} = \varphi$. Moreover, the operator $(dd^c w)^{n-m+1} \wedge \beta^{m-1} = 0$. We will prove an analog of this theorem on the maximality of the (m,ψ)-subharmonic measure.

Theorem 3.10. *Let $K \subset D$ be (m,ψ)-regular compact set and $\psi(z)$ be a continuous function in the compact set K. Then, the $\mathcal{P}_{(m,\psi)}-$measure is maximal in the open set $D \backslash K$, i.e.,*

$$(dd^c \omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0.$$

Proof. According to Theorem 3.3, $\omega(z,K,D,\psi) \equiv \omega^*(z,K,D,\psi) \in C(\bar{D})$ for any $z \in D$. We fix a ball $B \subset\subset D\backslash K$ and construct the following function

$$v(z) = \sup\{u(z) : u \in sh_m(B) \bigcap C(\bar{B}),\ u|_{\partial B} \leq \omega^*(K,D,\psi)|_{\partial B}\}.$$

Then $v \in sh_m(B) \bigcap C(\bar{B})$ and it is maximal in the ball B, i.e., $(dd^c v)^{n-m+1} \wedge \beta^{m-1} = 0$. Since $v(z) = \omega^*(z,K,D,\psi),\ \forall z \in \partial B$ and it is maximal in B, then

$$v(z) \geq \omega^*(z,K,D,\psi),\ \forall z \in B.$$

Let us define the following function

$$w(z) = \begin{cases} \omega^*(z,K,D,\psi), & z \in D\backslash B \\ v(z), & z \in B. \end{cases}$$

It can be easily seen that according to the above definition we have $w(z) \in sh_m(D) \bigcap C(D)$ and $w(z) \in \mathcal{U}(K,D,\psi)$. As a result,

$$w(z) \leq \omega^*(z,K,D,\psi),\ \forall z \in D.$$

Consequently, we have $v(z) = \omega^*(z,K,D,\psi),\ \forall z \in B$ and $(dd^c \omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0$ in B. From the arbitrariness of the ball $B \subset D\backslash K$, we can conclude that

$$(dd^c \omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} = 0$$

in the open set $D\backslash K$. □

4. The weighted capacity $\mathcal{P}_m(E,D,\psi)$ and (m,ψ)-capacity $C_m(K,D,\psi)$ of a condenser (K,D).

We introduce the weighted capacity value $\mathcal{P}_m(E,D,\psi)$ and (m,ψ)-capacity of the condenser (K,D) using the weighted $\mathcal{P}_{m,\psi)}-$measure.

Let $E\subset D$ be a set and $\psi(z)$ be a bounded, negative and continuous function on E. As mentioned above, we consider the case $\sup\limits_{z\in E}\psi(z)<0$ in constructing the function $\omega^*(z,E,D,\psi)$.

Definition 4.1. The quantity

$$\mathcal{P}_m(E,D,\psi)=-\int\limits_D\omega^*(z,E,D,\psi)dV$$

is called the $\mathcal{P}_{(m,\psi)}-$capacity of the set E with respect to D.

Similarly, $\mathcal{P}_m(E,D)$ capacity (case $\psi(z)=-1$) the function $\mathcal{P}_m(E,D,\psi)\geq 0$ and it satisfies monotonicity, countable subadditivity. Moreover, $\mathcal{P}_m(E,D,\psi)=0$ if and only if E is a m-polar set.

Definition 4.2. Let K be a compact subset of a strongly m-regular domain $D\subset\mathbb{C}^n$. Then the following quantity

$$C_m(K,D,\psi)=\inf\left\{\int\limits_D(dd^cu)^{n-m+1}\wedge\beta^{m-1}:u\in sh_m(D)\bigcap C(D),u|_K\leq\psi|_K,\ \underline{\lim}_{z\to\partial D}u(z)\geq 0\right\}$$

is called (m,ψ)-capacity of the condenser (K,D).

Note that, if $\psi\equiv-1$, then the weighted (m,ψ)-capacity $C_m(K,D,\psi)$ coincides with $C_m(K,D)$, $\;C_m(K,D,-1)=C_m(K,D)$.

In the study of weighted (m,ψ)-capacity $C_m(K,D,\psi)$ for simplicity, we assume that the weight function ψ is continuous, $\psi(z)\in C(E)$, although many of the properties proved below remain valid for the general case of $\psi(z)$.

The (m,ψ)-capacity $C_m(K,D,\psi)$ has the following properties.

Proposition 4.3. *a) if $K_1\subset K_2\subset D$, then $C_m(K_1,D,\psi)\leq C_m(K_2,D,\psi)$.*
b) if $\psi_1\leq\psi_2$, then $C_m(K,D,\psi_1)\geq C_m(K,D,\psi_2)$.

The proof of the monotonicity properties of the condenser $C_m(K,D,\psi)$ follows easily from its definition.

Proposition 4.4. *If $K\subset D$ is a (m,ψ)-regular compact, then*

$$C_m(K,D,\psi)=\int\limits_K(dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}.$$

Proof. $\omega^*(z,K,D,\psi)\in sh_m(D)\bigcap C(D)$ and according to Theorem 3.10,

$$\int\limits_K(dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\geq C_m(K,D,\psi).$$

Conversely, $\forall\varepsilon$, $0<2\varepsilon<-\max\limits_{z\in K}\psi(z)$ and for any function

$$u\in sh_m(D)\bigcap C(D),\ u|_K\leq\psi|_K,\ \underline{\lim}_{z\to\partial D}u(z)\geq 0,$$

the set $F=\{z\in D: u(z)<(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})\cdot\omega^*(z,K,D,\psi)-\varepsilon\}\subset\subset D$ is open. Therefore, according to the comparison principle (Theorem 1.5),

$$\int\limits_F (dd^c u)^{n-m+1}\wedge\beta^{m-1}\geq (1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_F (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}.$$

Since $K\subset F$ and $(dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}=0$ in $D\backslash K$, it follows that

$$(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}=$$

$$=(1+\frac{2\varepsilon}{\max\limits_{z\in K}\psi(z)})^{n-m+1}\cdot\int\limits_F (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq$$

$$\leq\int\limits_F (dd^c u)^{n-m+1}\wedge\beta^{m-1}\leq\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}.$$

The arbitrariness of $\varepsilon>0$ implies that

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}$$

and

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1}\wedge\beta^{m-1}\leq C_m(K,D,\psi).$$

□

Proposition 4.5. *For any compact $K\subset D$,*

$$C_m(K,D,\psi)=\inf\{C_m(E,D,\tilde\psi):\ E\supset K\},$$

where $\tilde\psi\in C(E)$, $\tilde\psi|_K=\psi|_K$ and E is $(m,\tilde\psi)-$regular compact in the domain D.

Proof. For any $\varepsilon>0$, there exists a function $u(z)$ with $u\in sh_m(D)\bigcap C(D), u|_K\leq\psi|_K$, $\varliminf\limits_{z\to\partial D}u(z)\geq 0$ such that the following inequality

$$\int\limits_D (dd^c u)^{n-m+1}\wedge\beta^{m-1}-C_m(K,D,\psi)<\varepsilon$$

holds. Since the function $\psi(z)$ is continuous on the compact set K, according to Whitney's theorem [8], there exists some continuous function $\tilde\psi(z)$ in D such that $\tilde\psi|_K=\psi|_K$. Then, $U=\{z\in D: u(z)<\tilde\psi(z)+\varepsilon\}$ is open, $U\supset K$. We take a $(m,\tilde\psi)-$regular compact E such that $E:\quad K\subset E\subset\subset U$ and consider the open set

$$F=\{z\in D: u(z)<(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde\psi(z)})\cdot\omega^*(z,E,D,\tilde\psi)-\varepsilon\}$$

where $0 < 2\varepsilon < -\max\limits_{z\in E}\tilde{\psi}(z)$. It is not difficult to check that $E \subset F \subset\subset D$. Applying the comparison principle again we have

$$C_m(E,D,\tilde{\psi}) = \int\limits_E \left(dd^c\omega^*(z,E,D,\tilde{\psi})\right)^{n-m+1} \wedge \beta^{m-1} =$$

$$= \int\limits_F \left(dd^c\omega^*(z,E,D,\tilde{\psi})\right)^{n-m+1} \wedge \beta^{m-1} \leq$$

$$\leq \frac{1}{\left(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \cdot \int\limits_F \left(dd^c u\right)^{n-m+1} \wedge \beta^{m-1} \leq$$

$$\leq \frac{1}{\left(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \int\limits_D \left(dd^c u\right)^{n-m+1} \wedge \beta^{m-1} <$$

$$< \frac{1}{\left(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} (C_m(K,D,\psi)+\varepsilon).$$

Thus,

$$C_m(K,D,\psi) \leq C_m(E,D,\tilde{\psi}) < \frac{1}{\left(1+\frac{2\varepsilon}{\max\limits_{z\in E}\tilde{\psi}(z)}\right)^{n-m+1}} \cdot (C_m(K,D,\psi)+\varepsilon).$$

The arbitrariness of $\varepsilon > 0$ implies that

$$C_m(K,D,\psi) = \inf\{C_m(E,D,\tilde{\psi}) : \ E \supset K\}.$$

□

Proposition 4.6. *If K is a (m,ψ)-regular compact, then*

$$C_m\left(K,D,\psi\right) = \sup\left\{\int\limits_K \left(dd^c u\right)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m\left(D\right) \bigcap C\left(D\right),\ \psi|_K \leq u|_K,\ \ u|_D < 0\right\}.$$

Proof. Since $C_m\left(K,D,\psi\right) = \int\limits_K \left(dd^c\omega^*\left(z,K,D,\psi\right)\right)^{n-m+1} \wedge \beta^{m-1}$ and

$$\psi|_K = \omega^*\left(z,K,D,\psi\right)|_K,\ \ \omega^*\left(z,K,D,\psi\right)|_D < 0,$$

it follows that

$$C_m\left(K,D,\psi\right) \leq \sup\left\{\int\limits_K \left(dd^c u\right)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m\left(D\right) \bigcap C\left(D\right),\ \psi|_K \leq u|_K,\ u|_D < 0\right\}.$$

On the other hand, for any function

$$u \in sh_m\left(D\right) \bigcap C\left(D\right),\ \psi|_K \leq u|_K,\ u|_D < 0, \forall z \in D,$$

we will construct a function v such that

$$v\left(z\right) = \max\left\{\left(1+\varepsilon\right)\omega^*\left(z,K,D,\psi\right),\ u\left(z\right)\right\},\ \varepsilon > 0.$$

Then we have

$$v \in sh_m(D) \bigcap C(D), \quad \psi|_K \le v|_K, \quad v|_D < 0, \quad v|_K = u|_K, \quad \lim_{z\to\partial D} v(z) = 0.$$

Let us now consider the following open set

$$F = \left\{ z \in D : \ (1+\varepsilon)\,\omega^*(z,K,D,\psi) + \varepsilon^2 < v(z) \right\},$$

where $0 < \varepsilon < -\max\limits_{z\in K} \psi(z)$. It is easy to check that $K \subset F \subset\subset D$.

Thus, according to the comparison principle and Theorem 3.10,

$$(1+\varepsilon)^{n-m+1} \int\limits_K [dd^c\omega^*(z,K,D,\psi)]^{n-m+1} \wedge \beta^{m-1} =$$

$$= (1+\varepsilon)^{n-m+1} \int\limits_F [dd^c\omega^*(z,K,D,\psi)]^{n-m+1} \wedge \beta^{m-1} \ge$$

$$\ge \int\limits_F (dd^c v)^{n-m+1} \wedge \beta^{m-1} \ge \int\limits_K (dd^c v)^{n-m+1} \wedge \beta^{m-1} = \int\limits_K (dd^c u)^{n-m+1} \wedge \beta^{m-1}.$$

The arbitrariness of $\varepsilon > 0$ implies that

$$\int\limits_K (dd^c\omega^*(z,K,D,\psi))^{n-m+1} \wedge \beta^{m-1} \ge \int\limits_K (dd^c u)^{n-m+1} \wedge \beta^{m-1}.$$

□

Definition 4.7. Let U be an open subset of D. The quantity

$$C_m(U,D,\psi) = \sup\{C_m(K,D,\psi) : K \subset U\}$$

is called (m,ψ)- capacity of the open set U.

It follows from the definition of the $(m,\psi)-$ capacity of the open set U and from proposition 4.5 that $C_m(U,D,\psi) = \sup\{C_m(K,D,\psi) : \ K \subset U\}$, where K is a $(m,\psi)-$regular compact. The monotonicity properties mentioned above for the compact $K \subset D$ also hold for an open set U. The proof methods used for the corresponding properties in [7] can be applied to prove propositions below.

Proposition 4.8. *If $U \subset D$ is an open set, then*

$$C_m(U,D,\psi) = \sup\left\{ \int\limits_U (dd^c u)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m(D) \bigcap C(D),\ \psi|_U \le u|_U,\ u|_D < 0 \right\} =$$

$$= \sup\left\{ \int\limits_U (dd^c u)^{n-m+1} \wedge \beta^{m-1} : u \in sh_m(D) \bigcap C^\infty(D),\ \psi|_U \le u|_U,\ u|_D < 0 \right\}.$$

Proposition 4.9. *For any increasing sequence of open sets $U_j \subset U_{j+1} \subset D$, $j \in \mathbb{N}$, we have*

$$C_m\left(\bigcup_j U_j, D, \psi\right) = \lim_{j\to\infty} C_m(U_j,D,\psi).$$

Now, we will define the (m,ψ) −external capacity of an arbitrary set $E \subset D$ by using the capacity of open sets.

Definition 4.10. The quantity

$$C_m^*(E,D,\psi) = \inf\{C_m(U,D,\psi): \ U \supset E\}$$

is called the (m,ψ)-external capacity of the set $E \subset D$, where $U \subset D$ is an open set in D.

Proposition 4.11. *(m,ψ) −external capacity is countably subadditive, i.e.*

$$C_m^*\left(\bigcup_j E_j, D, \psi\right) = \sum_j C_m^*\left(E_j, D, \psi\right), \ \ E_j \subset D, \forall j \in \mathbb{N}.$$

In [4], it is shown that for $\psi \in F$ and $E \subset\subset D$, there exist positive constants c_1 and c_2 such that the relation

$$c_1 C_m^*\left(E,D,\psi\right) \le \mathcal{P}_m\left(E,D,\psi\right) \le c_2 (C_m^*\left(E,D,\psi\right))^{\frac{1}{n-m+1}}$$

holds. Where

$$F = F\left(D\right) = \Big\{\varphi \in sh_m\left(D\right): \forall z^0 \in D, \ \exists \text{ a neighbourhood } U \ni z^0, \ \ \exists \varphi_j \in sh_m\left(D\right) \bigcap L^\infty\left(D\right),$$

$$\lim_{z\to\partial D} \varphi_j\left(z\right) = 0, \forall j \in \mathbb{N}, \ \ \varphi_j \searrow \varphi \text{ on } U, \ \sup_j \int\limits_D \left(dd^c \varphi_j\right)^{n-m+1} \wedge \beta^{m-1} < +\infty \Big\}.$$

From this, for $\psi \in F\left(D\right)$ and $E \subset\subset D$, we obtain the important result: $C_m^*\left(E,D,\psi\right) = 0$ if and only if E is m-polar.

The last fact, that the weighted external capacity $C_m^*\left(E,D,\psi\right) = 0$ if and only if E is m-polar, also follows from the following inequality.

Theorem 4.12. *The following inequality holds:*

$$\left(-\inf_{z\in E} \psi\left(z\right)\right)^{n-m+1} C_m^*\left(E,D\right) \ge C_m^*\left(E,D,\psi\right) \ge \left(-\sup_{z\in E} \psi\left(z\right)\right)^{n-m+1} C_m^*\left(E,D\right), \forall E \subset D. \tag{4.1}$$

Proof. From the definition of the (m,ψ) −external capacity, it suffices to prove the inequality (4.1) for (m,ψ) −regular compact sets.

Since $D \subset \mathbb{C}^n$ is a strongly m-regular domain, there exists a function $\rho \in sh_m\left(D\right) \bigcap C^2\left(D\right)$ such that $\rho|_D < 0$ and $\lim\limits_{z\to\partial D} \rho\left(z\right) = 0$. According to inequality (3.1), for any $\varepsilon > 0$ the following holds:

$$-\inf_{z\in E} \psi\left(z\right) \cdot \omega^*\left(z,E,D\right) + 2\varepsilon\rho\left(z\right) < \omega^*\left(z,E,D,\psi\right) + \varepsilon\rho\left(z\right) < -\sup_{z\in E} \psi\left(z\right) \cdot \omega^*\left(z,E,D\right), \ \forall z \in D.$$

The functions $\omega^*\left(z,E,D\right)$, $\rho\left(z\right)$ and $\omega^*\left(z,E,D,\psi\right)$ can be extended to a domain $G \supset\supset D$ such that the following inequality holds for the extended functions:

$$-\inf_{z\in E} \psi\left(z\right) \cdot \tilde{\omega}^*\left(z,E,D\right) + 2\varepsilon\tilde{\rho}\left(z\right) \ge \tilde{\omega}^*\left(z,E,D,\psi\right) + \varepsilon\tilde{\rho}\left(z\right) \ge -\sup_{z\in E} \psi\left(z\right) \cdot \tilde{\omega}^*\left(z,E,D\right), \ \ \forall z \in G\backslash D,$$

where, $\tilde{\omega}^*(z,E,D)$, $\tilde{\rho}(z)$ and $\tilde{\omega}^*(z,E,D,\psi)$ are, respectively, the m-subharmonic and continuous extensions of the functions $\omega^*(z,E,D)$, $\rho(z)$ and $\omega^*(z,E,D,\psi)$ to the domain G. Applying the comparison principe, we obtain:

$$\int_D \left[dd^c\left(-\inf_{z\in E}\psi(z)\,\omega^*(z,E,D)+2\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}\geq$$

$$\geq\int_D \left(dd^c\left(\omega^*(z,E,D,\psi)+\varepsilon\rho(z)\right)\right)^{n-m+1}\wedge\beta^{m-1}$$

$$\geq\int_D \left[dd^c\left(-\sup_{z\in E}\psi(z)\,\omega^*(z,E,D)\right)\right]^{n-m+1}\wedge\beta^{m-1}.$$

It is known that the integrals

$$\int_D \left[dd^c\left(-\inf_{z\in E}\psi(z)\,\omega^*(z,E,D)+2\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}$$

and

$$\int_D \left[dd^c\left(\omega^*(z,E,D,\psi)+\varepsilon\rho(z)\right)\right]^{n-m+1}\wedge\beta^{m-1}$$

can be expressed as,

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}\int_D \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_1$$

and

$$\int_D \left[dd^c\omega^*(z,E,D,\psi)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_2,$$

where C_1 and C_2 are positive constants. According to Theorem 3.10,

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}\int_E \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_1\geq$$

$$\geq\int_E \left(dd^c\omega^*(z,E,D,\psi)\right)^{n-m+1}\wedge\beta^{m-1}+\varepsilon C_2\geq$$

$$\geq\left(-\sup_{z\in E}\psi(z)\right)^{n-m+1}\int_E \left[dd^c\omega^*(z,E,D)\right]^{n-m+1}\wedge\beta^{m-1}.$$

Since $\varepsilon>0$ is arbitrary, we conclude that

$$\left(-\inf_{z\in E}\psi(z)\right)^{n-m+1}C_m^*(E,D)\geq C_m^*(E,D,\psi)\geq\left(-\sup_{z\in E}\psi(z)\right)^{n-m+1}C_m^*(E,D),\forall E\subset D.$$

□

Corollary 4.13. *$C_m^*(E,D,\psi)=0$ if and only if E is m-polar.*

Indeed, it is known that the external capacity $C_m^*(E,D)=0$ if and only if E is m-polar set in D. From inequality (4.1), it follows that Corollary 4.13 holds.

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