Variational analysis of averaged nonlinearity-managed solitons

J.S. Yuldashev[*,1,2]

[1]Physical-Technical Institute, Uzbek Academy of Sciences, Chingiz Aytmatov Street, 2-B, 100084 Tashkent, Uzbekistan
[2]National University of Uzbekistan, Department of Theoretical Physics, University Street, 4, 100174 Tashkent, Uzbekistan

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*Corresponding author: e-mail yoldoshevjasur95@gmail.com, Phone +998 99 791 27 95

Аналитически рассмотрено нелинейное уравнение Шредингера с периодически изменяющимся коэффициентом нелинейности, описывающее материальные волновые солитоны в конденсации Бозе-Эйнштейна (БЭК) и оптические солитоны. Вариационный подход применяется к полученному усредненному уравнению, которое описывает солитоны, управляемые нелинейностью. Показано хорошее совпадение аналитических решений, полученных с помощью вариационного анализа, с численными решениями полученного усредненного уравнения.

Ключевые слова: солитоны, нелинейное управление, вариационный метод.

The nonlinear Schrodinger equation with periodically varying nonlinearity coefficient that describes matter-wave soliton in in Bose-Einstein Condensation (BEC) and optical solitons is considered analytically. Variational approach is applied on the derived averaged equation which describes nonlinearity-managed solitons. A good agreement between analytical solutions obtained by means of variational analysis and numerical solutions of derived averaged equation is demonstrated.

Keywords: solitons, nonlinearity management, variational method.

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I. Introduction

After theoretically treatment (Korteweg-De Vries equation and it's solution) of the soliton (or solitary wave) phenomenon, first observed and described in 1834 by Jhon Scott Russel in Union canal, on the shallow water by D. Korteweg and G. de Vries [1], soliton phenomenon has been of fundamental importance in many fields of physics and mathematics. Solitons are localized waves that have been demonstrated theoretically and experimentally in various physics environments. For the last several decades researches devoted to soliton phenomenon have been mostly focused on nonlinear optics [2] and Bose-Einstein Condensation (BEC) [3]. Possibly, the notable application of solitons is signal transmission over long distances in the form of optical solitons in optic fiber communication. In optics, solitons are formed due to the balance between nonlinearity and dispersion/diffraction effects of the medium and governed by Nonlinear Schrodinger (NLS) equation. In the context of the BEC, the balance between dispersion due to the kinetic energy of atoms and the attractive mean field energy can lead to the formation of bright solitons. And governing equation is often referred to Gross-Pitaevskii equation (GPE). These equations are integrable and exact analytical soliton solutions can be found by inverse scattering method [4]. Conventional solitons are found homogeneous media with constant parameters. However, on many occasions, it is required to consider solitons in nonuniform media, or subjected to strong time modulation. Dispersion-managed solitons, which is important concept in fiber optic communication, is prominent example of former setting [5, 6]. Possible stabilization of matter-wave solitons using nonlinearity management, that can be realized by time-periodic variation of scattering length which is called Feshbach resonance, is the another main reason for studying solitons with time varying parameters. Namely, stabilization of bright solitons is observed in two-dimensional BEC in [7] and two- and three-dimensional BEC in [8, 9].

Analytical approximations of dispersion and nonlinearity-managed solitons can be obtained from the corresponding averaged equations. For the former case, averaged NLS equation was derived in [10]. Dynamics of discrete solitons was considered in the periodically varying nonlinear media [11, 12]. Four equivalent methods for averaging NLS equation with nonlinearity management were introduced in [13].

In this study, variational approximation method, firstly proposed for bright optical solitons in [14], is applied on the derived averaged NLS equation ([13]) to formulate stationary soliton solution of NLS equation.

In the next section, the main model and its averaged form will be introduced. Variational analysis of the averaged model is considered in section **III** and confirmation of analytical predictions and a relationship between nonlinear coefficient and real physical quantities of Feshbach resonance will be demonstrated in section **IV** by the results of numerical analysis.

II. The model

The main one-dimensional (1D) model for the Feshbach resonance, which is used to control Bose-Einstein condensates by changing external magnetic field that leads to variation of scattering length of interatomic interactions, is given by GPE with variable nonlinearity coefficient:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi(x,t)+\frac{g(t)N}{2\pi a_\perp^2}|\Psi|^2\Psi(x,t),$$

where $\Psi(x,t)$ is complex atomic envelope field and normalized by requiring

$$\int|\Psi(x,t)|^2dx=1,$$

Here, t is time, x is spatial coordinate, m is the atomic mass, N is the total number of atoms in the condensate, $\hbar$ is the Plank constant and $a_\perp=\sqrt{\hbar/m\omega_\perp}$ is oscillator length with $\omega_\perp$ being confining frequency in the transverse direction. The interaction between atoms in the condensate is described by the parameter $g(t)=(4\pi\hbar^2/m_a)a_s(t)$ with a_s the scattering length. By introducing new scaled variables

$$\tilde{t}=\frac{t}{t_s},\quad \tilde{x}=\frac{x}{a_\perp},\quad \tilde{\psi}(\tilde{x},\tilde{t})=\sqrt{a_\perp}\Psi(x,t),$$

where $t_s=2ma_\perp^2/\hbar$ is dimensionless time units, we get the following dimensionless sequation

$$\frac{i\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\gamma(t)|\psi|^2\psi=0, \qquad (1)$$

here $\gamma(t)=4Na_s(t)/a_\perp=\gamma_0+\gamma_1(t)=\gamma_0+\gamma_1\cos(\omega t)$. For notational convenience, here and hereafter tilde-symbols are omitted. The case $\gamma_1\sim\omega\sim1/э$, $э\ll1$ is referred to strong management, while the case $\gamma_1\sim0(1)$, $\omega\sim1/э$ is referred to weak management. By introducing the transformation $\psi(x,t)=u(x,t)\exp[i\Gamma(t)|u(x,t)|^2]$, following averaged equation of Eq.(1)

$$iu_t+u_{xx}+\gamma_0|u|^2u+\sigma(|u|_x^2)^2+2|u|^2|u|_{xx}^2|u| \qquad (2)$$

and its averaged Hamiltonian form:

$$\bar{H}=\int_R(|u_x|^2-\frac{1}{2}\gamma_0|u|^4+\sigma|u|^2(|u|_x^2)^2)$$

are derived in [9], where $\Gamma_1(t)=\gamma_1(t)$ and $\sigma=\left\langle\Gamma^2(t)\right\rangle=\frac{1}{2}\left(\frac{\gamma_1}{\omega}\right)^2$.

III. A variational formulation of the averaged NLS equation

In this section we consider approximation analysis of averaged Eq. (2). Variational approximation method on the averaged equation is applied. We start from restating averaged NLS equation (2) as a variational problem corresponding to the Lagrangian density (LD) given by

$$L=\frac{i}{2}\left(u\frac{\partial u^*}{\partial t}-u^*\frac{\partial u}{\partial t}\right)+|u_x|^2+\sigma|u|^2(|u|_x^2)^2-\frac{\gamma_0}{2}|u|^4,$$

and with definition of Lagrangian variation

$$\frac{\delta L}{\delta u^*}=\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t^*}+\frac{\partial}{\partial x}\frac{\partial L}{\partial u_x^*}-\frac{\partial L}{\partial u^*},$$

where asterisk denotes the complex conjugate. We use the trial secant function as ansatz function with real time-dependent amplitude $A(t)$, width $a(t)$, chirp $b(t)$ and phase $\phi(t)$.

$$u(x,t)=A(t)\operatorname{sech}\left(\frac{x}{a(t)}\right)\exp[ib(t)x^2+i\varphi(t)], \qquad (3)$$

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A set of coupled ordinary differential equations for the ansatz parameters (A, a, b, ϕ) can be obtained by the following variational principle:

$$\delta\int \bar{L}dx = 0\,, \tag{4}$$

where $\bar{L}$ denotes averaged Lagrangian density which is obtained by integrating the result of inserting the secant function given by (3) into the Lagrangian density. By performing integration, we obtain

$$\bar{L} = \frac{\pi^2}{6}A^2a^3b_t + 2A^2a\varphi_t + \frac{2A^2}{3a} + \frac{2\pi^2}{3}A^2a^3b^2 + \\ + \frac{64\sigma A^6}{35a} - \frac{2\gamma_0}{3}A^4a.$$

Using Eq. (4) for the secant ansatz parameters, we obtain following system of equations

$$\frac{\pi^2}{3}a^3b_t + 4a\varphi_t + \frac{4}{3a} + \frac{4\pi^2}{3}a^3b^2 + \frac{128\sigma A^4}{35a} - \\ -\frac{8\gamma_0}{3}A^4a = 0, \tag{5}$$

$$\frac{\pi^2}{2}a^2b_t + 2\varphi_t - \frac{2}{3a^2} + 2\pi^2a^2b^2 - \frac{64\sigma A^4}{105a^2} - \\ -\frac{2\gamma_0}{3}A^2 = 0, \tag{6}$$

$$\frac{\partial}{\partial t}\left(\frac{\pi^2}{6}A^2a^3\right) - \frac{4\pi^2}{3}A^2a^3b = 0\,, \tag{7}$$

$$\frac{\partial}{\partial t}(2A^2a) = 0 \tag{8}$$

for the amplitude, width, chirp, and phase, respectively. Conservation of number of atomic population (pulse/beam energy in the case of optics) $2A^2a = N = const$, also can be obtained from the well-known invariant of the NLS equation $\int_{-\infty}^{+\infty}|u(x,t)|^2dx = const$, is followed from Eq. (8). Using the fact that $2A^2a = N$ is constant, we obtain from Eq. (7)

$$a_t = 4ba. \tag{9}$$

By comparing Eqs. (5) and (6) we get

$$ab_t - \frac{4}{\pi^2}\frac{1}{a^3} + 4b^2a - \frac{64\sigma N^2}{35\pi^2a^5} + \frac{\gamma_0N}{\pi^2a^2} = 0,$$

which can be combined with Eq.(9) to obtain equation for only width

$$a_{tt} = \frac{16}{\pi^2a^3} + \frac{256\sigma N^2}{35\pi^2a^5} - \frac{4\gamma_0N}{\pi^2a^5}\,. \tag{10}$$

The last equation can be considered as analogues to that of particle moving in a potential field which is given by

$$P(a) = -\frac{8}{\pi^2}\frac{1}{a^2} + \frac{64\sigma N^2}{35\pi^2a^4} - \frac{4\gamma_0N}{\pi^2a}.$$

Stationary solution of averaged model (2) can be found by setting $da/dt = 0$ (fixed point a_c) in the equation (10), which corresponds to minimum of the potential given by

$$\frac{dP(a)}{da} = \frac{16}{\pi^2a^3} - \frac{256\sigma N^2}{35\pi^2a^5} + \frac{4\gamma_0N}{\pi^2a^2} = 0.$$

And the fixed point is the solution of following algebraic cubic equation:

$$35\gamma_0Na_c^3 - 140a_c^2 - 64\sigma N^2 = 0. \tag{11}$$

Equation (11) has an analytical solution in the following form:

$$a_c = \frac{1}{105\gamma_0N}\left(140 - C - \frac{140^2}{C}\right), \tag{12}$$

where

$$C = 100\sqrt[3]{\frac{-\Delta_1 \pm \sqrt{\Delta_1^2 - 4\times1.4^6}}{2}}$$

with $\Delta_1 = (5.488 + 2.1168\sigma\gamma_0^2N^4)$. The amplitude ($A_c$) corresponds to width of stationary solution (a_c) found directly from Eq. (8), that is

$$A_c = \sqrt{N/2a_c}\,. \tag{13}$$

IV. Comparison of variational and numerical analysis

In the original model, the cosinusoidal function $\gamma(t) = \gamma_0 + \gamma_1\cos(2\pi t)$ is used, in which case $\sigma = \gamma_1^2/8\pi^2$. Numerical calculations are carried out and the results have been compared with variational analysis. The Newton-Conjugate-Gradient method [15] is used to find stationary solution of Eq. (2) and it has been found that a variational analysis has a very good agreement with numerical results. In Fig. 1, the minimum of potential (red dot) and corresponding width (a_c = 1.5639) is shown, while Fig. 2 demonstrates the comparison of numerical solution with variational solution corresponding to that minimum of potential in the case of N = 6.3753, γ_0 = 0.5, and γ_1 = 1.6.

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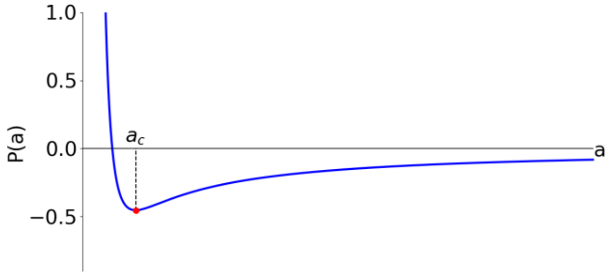

Figure 1. (Color online)The potential and fixed point (a_c = 1.5639) corresponds to minimum of the potential in the case of following modulation parameters: γ_0 = 0.5, γ_1 = 1.6 and $\omega = 2\pi$.

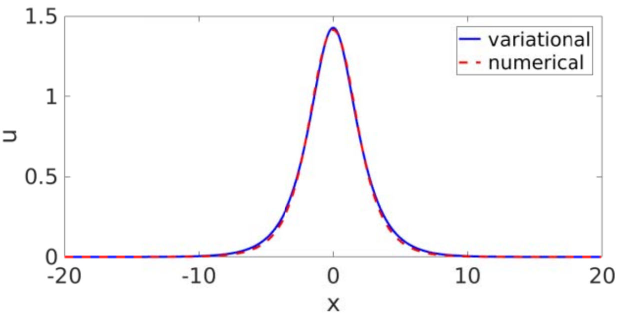

Figure 2. (Color online) Comparison of variational and numerical solutions of equation (2). Solid (red) line corresponds to variational solution while dashed (blue) line shows the numerical solution. Parameters are taken the same as in Fig. 1.

Analytical solutions for the soliton amplitude (A) and width (a) show that the amplitude of the soliton increases as the number of atoms (N) increases, as in the case without modulation. However, the width of the soliton initially decreases as the norm increases, and then starts increasing after reaching a certain point unlike in the case without modulation it only decreases (Fig. 3). And these analytical results are confirmed by the numerical analysis. In Fig. 3 dependence of the soliton's width and amplitude on the number of atoms is shown in the case of γ_0 = 1, γ_1 = 1.2.

Since physical time and length are scaled with characteristic time (t_s) and length ($a_\perp$) units, actual physical quantities can be found by multiplying the dimensionless time and length, which are obtained numerically or analytically, by the time and length units. The values of the nonlinear parameters γ_0 and γ_1 may change as magnetic field B varies [16]. The relationship between nonlinear coefficients (γ_0 and γ_1) and external magnetic field (B_0 and B_1) is illuminated in detail in the appendix. For the isotope of rubidium

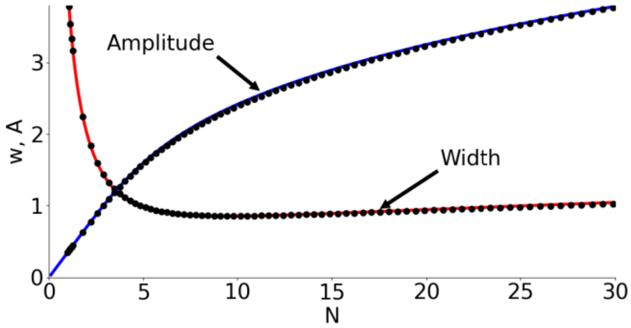

Figure 3. (Color online) Dependences of the soliton width and amplitude. Solid (red and blue) lines corresponds to width and amplitude (respectively) found analytically from Eqs. (12) and (13). Dots (black) represent the numerically found soliton widths and amplitudes. Parameters: γ_0 = 1, γ_1 = 1.2.

(^{85}Rb) whose mass is m = 84.9118 amu, scattering length is a_s = 0.8 nm in a magnetic field B ≈65 μT. To create 1D (cigar-shaped BEC) condensate, an external magnetic trap may be assumed as highly anisotropic characterized by confining axial $\omega_\parallel = 7.2\pi$ Hz and transverse $\omega_\perp = 720\pi$ Hz frequencies. Accordingly, for the ^{85}Rb condensate with N = 200000 atoms, the time and space units are 88.4 ms and 0.575 μm, repectively.

V. Conclusion

Overall in this study, the averaged equation (2) that governs the nonlinearity management in nonlinear optics and BEC has been solved by using semi-analytical variational method and several stationary solutions have been found. To verify the validity of the analytical results, numerical analysis has been carried out and it is found that the results obtained by variational approach corroborate with numerical results.

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Appendix

Now, we find the relationship between normalized nonlinear coefficient and real quantities of external magnetic field. In the original model (1), the nonlinear coefficient $\gamma(t)$ is related to the atomic scattering length (a_s) as following

$$\gamma(t) = \gamma_0 + \gamma_1 \cos(\omega t) = \frac{4\pi N a_s(t)}{a_\perp}. \quad \text{(A.1)}$$

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Here, m is the mass of atoms in condensate. The atomic scattering length can be changed both in the value and sign by a variation of an external magnetic field. The atomic scattering length depends on the external magnetic field in the following resonant form:

$$a_s(t) = a_B\left(1 - \frac{D}{B(t) - B_r}\right), \qquad \text{(A.2)}$$

where D is the width of the resonance, B_r is the position where resonance occurs, a_B is the background value of the scattering length and $B(t)$ is external magnetic field that can be changed in the form of $B_0 + B_1\cos(\omega t)$. By expanding the left-hand side of the relation (A.2) into Fourier series, we can get first approximation (α_0 and α_1) of scattering length, that is

$$a_s(t) = a_B\left(1 - \frac{D}{B(t) - B_r}\right) \approx \alpha_0 + \alpha_1\cos(\omega t).$$

By integrating Fourier coefficients and comparing with relation (A.1), one can obtain relationship between modulation parameters and real physical quantities as follows:

$$\gamma_0 = \frac{4\hbar^2}{m}\alpha_0 \text{ and } \gamma_1 = \frac{4\hbar^2}{m}\alpha_1,$$

where $\alpha_0 = a_B\left(1 - \frac{D}{\sqrt{(B_0 - B_r)^2 - B_1}}\right)$ and

$$\alpha_1 = \frac{2Da_B}{B_1}\left(\frac{B_0 - B_r}{\sqrt{(B_0 - B_r)^2 - B_1}} - 1\right)$$

are Fourier coefficients.

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Ўртачалашган ночизиқлилиги бошқариладиган солитонларнинг вариацион анализи

Ж.С. Юлдашев

[1]Ўзбекистон Фанлар Академияси Физика-техника институти. Чингиз Айтматов кўч. 2Б, 100084, Тошкент, Ўзбекистон
[2]Ўзбекистон Миллий университети, Университет кўч., 4, 100176, Тошкент, Ўзбекистон

Бозе-Эйнштейн конденсацияда модда-тўлқинли солитонлар ва оптик солитонларни тавсифловчи даврий равишла ўзгарувчан ночизиқли коэффициентли ночизиқлиқ Шредингер тенгламаси аналитик тарзда ўрганиб чиқилади. Ночизиқли бошқариладиган солитонларни тавсифловчи олинган ўртача тенгламага вариацион методи қўлланилади. Вариацион анализ ёрдамида олинган аналитик ечимлар ва олинган ўртача тенгламанинг сонли ечимларининг мос келиши кўрсатилди.

Калит сўзлар: солитонлар, ночизиқлиқ бошқарищ, вариацион методи.

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