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FOKAS METHOD FOR THE HEAT EQUATION ON METRIC GRAPHS

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Abstract. The paper presents a method for constructing solutions to initial-boundary value problems for the heat equation on simple metric graphs such as a star-shaped graph, a tree, and a triangle with three converging edges. The solutions to the problems are constructed by the so-called *Fokas method*, which is a generalization of the Fourier transform method. In this case, the problem is reduced to a system of algebraic equations for the Fourier transform of the unknown values of the solution at the vertices of the graph.

Keywords: initial-boundary value problem, heat equation, Fokas method, metric graph.

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CONTENTS

1. Introduction

It is known that boundary-value problems for partial differential equations on metric graphs are used to model diffusion processes and wave propagation in branched structures in many problems of physics, chemistry and biology [2, 4]. With that the generalized condition of the local flow preservation, i.e., the Kirchhoff condition, should hold at the branching points of the graph. In the simplest case this condition consists of the continuity condition for the solution at the vertex of the graph and the equality to zero of the sum of the outcoming one-sided derivatives at the vertex.

The graph consists of points called *vertices* and segments, whose ends lay at the vertices of the graph [3]. These segments are called the *edges* of the graph. We will consider only connected graphs, i.e., the graphs, from each point of which one can get to the other by the edges of the graph. By doing the isometric matching of each edge of graph to a number interval we obtain the so-called *metric graph*. The metric here is understood in the sense of the geodesic metric, i.e., as the shortest distance by the graph edges between two points of the graph. It is clear that such structure is one-dimensional, while the realistic branched structures are two-dimensional (the graphene nanotubes, for instance) or three-dimensional. Thus, there emerges the expediency question of modeling the real problems in engineering and physics by the use of metric graphs. In [5, 18], a boundary-value problem for the stationary Schrödinger equation in thin (with small transverse sections) branched domains was studied. It was proven that in the case where the diameter of the transverse section tends to zero, the solution of the considered problem tends to the solution of the boundary-value problem on the corresponding quantum (metric) graph. With that from the Schrödinger equation in the small

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domain of the branching point, which becomes the graph vertex in the limit, the Kirchhoff conditions are obtained. This result holds also in the case of the heat equation since the heat equation becomes the stationary Schrödinger equation under the Laplace transformation in time.

Another justification of the modeling of the heat process in the branched domains by use of metric graphs can be found in [16]. Also there was used the Fokas method to obtain the solution to a boundary-value problem for the heat equation on the compact star graph.

In the present paper we generalize the method of the unified Fokas transformation [6–8, 15–17] for the solution of the heat equation onto the case of some other metric graphs. The main aim of the paper is to demonstrate the advantage and simplicity of the usage of the aforementioned method for the solution of initial-boundary value problems of metric graphs. In the beginning we give the statement of the problem and some preliminary results obtained in [12, 15, 17]. In the following sections we give our new results. We give the method of the construction of the solution of initial-boundary value problems for the heat equation on tree graphs and a graph with a cycle consisting of triangular part and having additional edges coming from the vertices. Note that the solvability of the initial-boundary value problems on metric graphs is studied in [14, 19].

2. Statement of the Problem and Preliminary Results

Let $\Gamma = E \cup V$ be a connected metric graph, where $E = \{b_j\}_{j=1}^n$ is the set of its edges, and $V = \{\nu_j\}_{j=1}^m$ is the set of vertices [3]. Let us define the coordinates x_j on the edges of the graphs via the isometric mapping of these edges onto the intervals $(0, L_j)$, $j = 1, 2, \ldots, n$.

We say that the vertex ν *touches* the edge b_j if it is the end of this edge, and denote it by $b_j \sim \nu$. The number of the elements of the set $\{b : b \sim \nu, b \in E\}$ is called the *valence* of the vertex ν. If the valence of the vertex is equal to one, then it is called *boundary*. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_{m_1}\} = \partial\Gamma \subset V$ be the boundary vertices of the graph. Without loss of generality we will further use x instead of x_j.

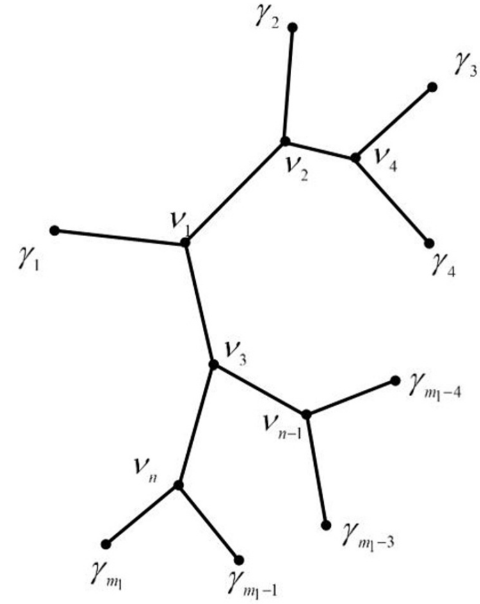

Fig. 2.1. Metric graph Γ

Consider the heat equation

$$u_t^{(j)}(x,t) = u_{xx}^{(j)}(x,t), \quad x \in b_j, \quad t > 0, \quad j = 1, 2, \ldots, n \tag{2.1}$$

on each edge of the graph.

We claim that the solution of Eq. (2.1) satisfies the initial condition

$$u^{(j)}(x,0) = u_0^{(j)}(x), \quad j = 1, 2, \dots, n. \tag{2.2}$$

In the branching points (i.e., in the internal vertices) of the graph the solution must satisfy the following *gluing conditions*:

(a) the values at the vertex ν of all functions $u^{(j)}$, for which $b_j \sim \nu$, are the same;

(b) the sum of one-sided derivatives at each vertex ν of all functions $u^{(j)}$, for which $b_j \sim \nu$, is equal to zero:

$$\sum_{b_j \sim \nu} \left. \frac{\partial u^{(j)}(x,t)}{\partial x} \right|_{\nu} = 0, \;\; \nu \in V \backslash \partial\Gamma, \;\; t \in [0,\, T]. \tag{2.3}$$

The first of these equations is called the *condition of the continuity* of the solution at the vertex, and the second — the *condition of flow preservation.* These conditions are also called the *Kirchhoff conditions*, or *conditions of δ type* at the vertex.

We claim that the following boundary conditions hold at the boundary vertices of the graph:

$$\begin{gathered} u^{(j)}(x,t)|_{\nu} = g_0^{(j)}(t), \text{ if } x_j = 0 \text{ at the vertex } \nu, \\ u^{(j)}(x,t)|_{\nu} = h_0^{(j)}(t), \text{ if } x_j = L_j \text{ at the vertex } \nu, \;\; b_j \sim \nu. \end{gathered} \tag{2.4}$$

The initial data $u_0^{(j)}$, $j = \overline{1, n}$, satisfy the gluing conditions at the nonboundary vertices of the graph and the matching conditions at the boundary vertices.

Further in this section we consider a few examples of the usage of the method of the unified Fokas transformation. Let us start with the simple case considered in [8].

2.1. Heat equation on a half-line. Consider an initial-boundary value problem for Eq. (2.1) on a half-line (see [8]). We need to find the regular solution of the equation $u_t = u_{xx} + f(x,t)$ in the domain $E = \{(x,t) : x > 0, 0 < t \le T\}$, $T > 0$, continuous in the closure of this domain and satisfying the initial condition $u(x,0) = u_0(x)$, $x \ge 0$, the boundary condition $u(0,t) = g_0(t)$ and the asymptotic condition $\lim\limits_{x \to \infty} u(x,t) = 0$, $0 \le t \le T$. We suppose that the initial and boundary data are sufficiently smooth functions, and $u_0(0) = g_0(0)$, $f(x,t) \in C^1(\bar{E})$, the functions $f(x,t)$ and $u_0(x)$ tend to zero as $x \to +\infty$, are absolutely integrable functions on $(0, +\infty)$ in x for all $0 \le t \le T$ and have bounded derivatives in their domains.

The Fokas method consists of three stages. The first stage is identical to the procedure used in the realization of the regular transformations. The second one requires the Cauchy theorem and Jordan lemma. The third one includes only the algebraic transformations.

1. We call the *global relation* an equation which connects an unknown solution with its values and the values of its derivatives at the boundary of the domain.

In our case this relation is obtained by multiplication of the equation by $e^{i(\lambda x + wt)}$, integration by the domain and usage of Green formula. Here the diffuse relation has the form $w = \lambda^2$. After simple transformations we easily obtain

$$e^{\lambda^2 t}\widehat{u}(-i\lambda, t) = \widehat{u}_0(-i\lambda) - \widetilde{g}_1(\lambda^2, t) - i\lambda\widetilde{g}_0(\lambda^2, t), \quad \operatorname{Im}\lambda \le 0, \tag{2.5}$$

where

$$\widehat{u}(-i\lambda, t) = \int\limits_0^\infty e^{-i\lambda x} u(x,t)dx, \quad 0 \le t \le T, \quad \operatorname{Im}\lambda \le 0, \tag{2.6}$$

$$\widehat{u}_0(-i\lambda) = \int\limits_0^\infty e^{-i\lambda x} u_0(x)dx, \quad \operatorname{Im}\lambda \le 0, \tag{2.7}$$

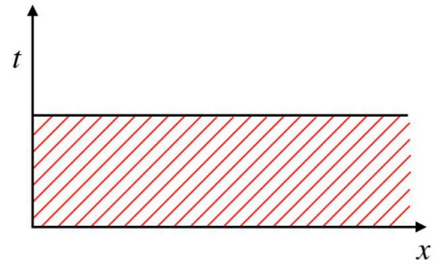

Fig. 2.2

$$\widetilde{g}_0(\lambda,t)=\int\limits_0^t e^{\lambda\tau}g_0(\tau)d\tau,\widetilde{g}_1(\lambda,t)=\int\limits_0^t e^{\lambda\tau}g_1(\tau)d\tau,\quad 0\le t\le T,\quad \lambda\in\mathbb{C},\tag{2.8}$$

$$g_1(t)=u_x(0,t),\quad g_0(t)=u(0,t),\quad 0<t\le T.\tag{2.9}$$

Note that

$$\left|e^{-i\lambda x}\right|=\left|e^{-i\lambda_R x+\lambda_I x}\right|=e^{\lambda_I x}.$$

Thus, these relations are bounded as $x\to\infty$ for all $\lambda_I\le 0$.

The functions $\widetilde{g}_0$ and $\widetilde{g}_1$ are defined for all complex values λ, while $\widehat{u}$ and $\widehat{u}_0$ are defined for $\operatorname{Im}\lambda\le 0$, hence the global relation (2.5) holds for $\operatorname{Im}\lambda\le 0$.

Note that the global relation (2.5) can also be obtained using the Laplace transformation defined by Eq. (2.6).

2. At this stage we need to obtain the integral representation of the solution.

For the heat equation defined on a half-axis this integral representation has the form

$$u(x,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{i\lambda x-\lambda^2 t}\widehat{u}_0(-i\lambda)d\lambda-\frac{1}{2\pi}\int\limits_{\partial D^+}e^{i\lambda x-\lambda^2 t}\left[\widetilde{g}_1(\lambda^2,t)+i\lambda\widetilde{g}_0(\lambda^2,t)\right]d\lambda,\tag{2.10}$$

where the contour ∂D^+ is the boundary of the domain D^+ defined as $D^+=\{\operatorname{Im}\lambda\ge 0,\ \operatorname{Re}\lambda^2<0\}$ (see Fig. 2.3).

Indeed, solving the global relation (2.5) with respect to $\widehat{u}(-i\lambda,t)$, and then using the inverse Fourier transformation, we find the expression similar to Eq. (2.10), but with the integration contour along the real axis instead of ∂D^+. In order to pass from the integral on the real axis to ∂D^+, we use the Cauchy theorem and Jordan lemma.

Firstly we consider the function

$$e^{i\lambda x-\lambda^2 t}\widetilde{g}_1(\lambda^2,t)=e^{i\lambda x}\int\limits_0^t e^{-\lambda^2(t-\tau)}g_1(\tau)d\tau,$$

which is a holomorph function of λ. The latter equality contains two exponents

$$e^{i\lambda x}=e^{i\lambda_R x-\lambda_I x},\quad e^{-\lambda^2(t-\tau)}=e^{-\operatorname{Re}(\lambda^2)(t-\tau)-i\operatorname{Im}(\lambda^2)(t-\tau)}.$$

Since $x\ge 0$ and $t-\tau\ge 0$, the exponents above are bounded as $\operatorname{Im}\lambda\ge 0$ and $\operatorname{Re}\lambda^2\ge 0$. Moreover, integrating by parts under these conditions one can easily obtain that the function above has the order

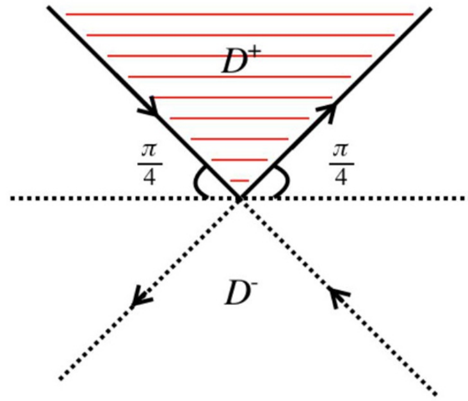

Fig. 2.3

$O\Big(\frac{1}{\lambda^2}\Big)$ as $\lambda \to \infty$:

$$e^{-\lambda^2 t} \int_0^t e^{\lambda^2 \tau} g_1(\tau) d\tau \sim \frac{g_1(t)}{\lambda^2}, \quad \lambda \to \infty.$$

Thus, from the Cauchy theorem for the domain bounded by the line $|\operatorname{Re}(\lambda)| = R$ and ∂D^+, using the Jordan lemma as $R \to +\infty$ it easily follows that the integral on the real line can be replaced with the contour integral on ∂D^+.

An analogical statements hold also for the integral of the function $i\lambda e^{i\lambda x - \lambda^2 t} \widetilde{g}_0(\lambda^2, t)$.

3. At this stage using the given boundary conditions, the global relation and also certain invariant transformations, we need to exclude the unknown boundary conditions from the integral representation above.

Our global relation is defined in $\operatorname{Im}\lambda \le 0$, but also we need to find $\widetilde{g}_1$ as $\lambda \in \partial D^+$, where $\operatorname{Im}\lambda \ge 0$.

Note that the substitution of λ with $-\lambda$ has two following properties: firstly, the set $\operatorname{Im}\lambda \le 0$ passes into the set $\operatorname{Im}\lambda \ge 0$ under this mapping, and secondly, the expressions $\widetilde{g}_0(\lambda^2, t)$ and $\widetilde{g}_1(\lambda^2, t)$ do not change under this mapping. Using this mapping, we obtain

$$\widetilde{g}_1(\lambda^2, t) = i\lambda \widetilde{g}_0(\lambda^2, t) + \widehat{u}_0(i\lambda) - e^{\lambda^2 t} \widehat{u}(i\lambda, t), \quad \operatorname{Im}\lambda \ge 0. \tag{2.11}$$

Substituting the term $e^{\lambda^2 t} \widehat{u}(i\lambda, t)$ into the integral representation (2.10), we obtain

$$-\frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x} \widehat{u}(i\lambda, t) d\lambda, \quad 0 < x < \infty, \quad t > 0.$$

The functions $e^{\lambda x}$ and $\widehat{u}(i\lambda, t)$ are bounded and holomorph in the upper half-plane, and

$$\widehat{u}(i\lambda, t) = \int_0^\infty e^{i\lambda x} u(x, t) dx \sim -\frac{u(0, t)}{i\lambda}, \quad \lambda \to \infty.$$

Then, by the Cauchy theorem and the Jordan lemma this integral is equal to zero.

Thus, from Eqs. (2.10) and (2.11) we obtain the solution of our problem as

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \widehat{u}_0(-i\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[2i\lambda \widetilde{g}_0(\lambda^2, t) + \widehat{u}_0(i\lambda) \right] d\lambda. \tag{2.12}$$

2.2. Solution of the heat equation on the finite interval by Fokas method. Let us now show the application of the Fokas method to the heat equation on the finite interval $0 < x < L$. Let us now implement the stages 1, 2 and 3 given in the previous section.

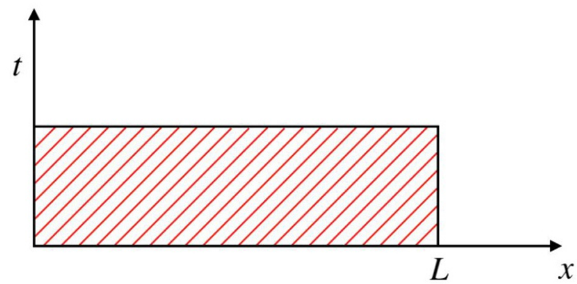

Fig. 2.4

1. The analogical actions as above give the following global relation:

$$e^{\lambda^2 t}\widehat{u}(-i\lambda,t)=\widehat{u}_0(-i\lambda)-\widetilde{g}_1(\lambda^2,t)-i\lambda\widetilde{g}_0(\lambda^2,t)+e^{-i\lambda L}\left[\widetilde{h}_1(\lambda^2,t)+i\lambda\widetilde{h}_0(\lambda^2,t)\right],\quad \lambda\in\mathbb{C},\tag{2.13}$$

where $\widehat{u}$ and $\widehat{u}_0$ are the "Fourier transformations on the finite interval" [1] for the functions $u(x,t)$ and $u_0(x)$ defined as

$$\widehat{u}(-i\lambda,t)=\int_0^L e^{-i\lambda x}u(x,t)dx,\quad \widehat{u}_0(-i\lambda)=\int_0^L e^{-i\lambda x}u_0(x)dx,\quad \lambda\in\mathbb{C},\tag{2.14}$$

$\widetilde{g}_1$, $\widetilde{g}_0$ are defined in Eq. (2.9), and $\widetilde{h}_1$ and $\widetilde{h}_0$ are defined as

$$\widetilde{h}_0(\lambda,t)=\int_0^t e^{\lambda\tau}h_0(\tau)d\tau,\quad \widetilde{h}_1(\lambda,t)=\int_0^t e^{\lambda\tau}h_1(\tau)d\tau,\quad t>0,\quad \lambda\in\mathbb{C},\tag{2.15}$$

where $h_0(t)=u(L,t)$, $h_1(t)=u_x(L,t)$, $t>0$.

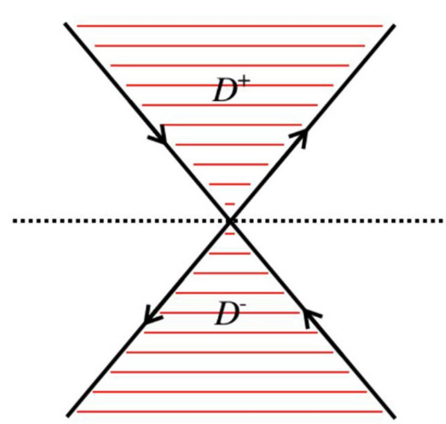

Fig. 2.5. The domains D^+ and D^- for the heat equation

2. From the relation (2.13) for $\widehat{u}(-i\lambda,t)$ using the inverse Fourier transformation (see [7, Sec. 6–8], [1, Sec. 7]), and also passing from the integral on the real axis to the integral by contour ∂D^+ in the integral including $\widetilde{g}_1$, $\widetilde{g}_0$, and to the integral by contour ∂D^- in the integral including $\widetilde{h}_1$, $\widetilde{h}_0$, we find that

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\widehat{u}_0(-i\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[\widetilde{g}_1 + i\lambda\widetilde{g}_0\right]d\lambda$$

$$-\frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}\left[\widetilde{h}_1 + i\lambda\widetilde{h}_0\right]d\lambda, \tag{2.16}$$

where D^- is the domain symmetrical to D^+ with respect to the real axis (see Fig. 2.5).

3. The transformation $\lambda \to -\lambda$ along with the global relation (2.13) gives two equations. Since there exist four unknown boundary values (two on each end of the domain), we require two boundary conditions. It is known that by the theory of the initial-boundary value problems for the partial differential equations these boundary conditions can be set in various ways.

As an example consider the first boundary-value problem with the boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = h_0(t), \quad t \geq 0.$$

We rewrite the global relation (2.13) in the form

$$e^{\lambda^2 t}\widehat{u}(-i\lambda,t) = G(\lambda,t) - \widetilde{g}_1 + e^{-i\lambda L}\widetilde{h}_1, \tag{2.17}$$

where G is an unknown function defined as

$$G(\lambda,t) = \widehat{u}_0(-i\lambda) - i\lambda\widetilde{g}_0(\lambda^2,t) + i\lambda e^{-i\lambda L}\widetilde{h}_0(\lambda^2,t). \tag{2.18}$$

Substituting in Eq. (2.17) λ with $-\lambda$, we obtain

$$e^{\lambda^2 t}\widehat{u}(i\lambda,t) = G(-\lambda,t) - \widetilde{g}_1 + e^{i\lambda L}\widetilde{h}_1. \tag{2.19}$$

From Eqs. (2.17) and (2.19) we obtain

$$\widetilde{g}_1 = \frac{1}{e^{i\lambda L}-e^{-i\lambda L}}\left\{e^{i\lambda L}G(\lambda,t) - e^{-i\lambda L}G(-\lambda,t) + e^{\lambda^2 t}\left[e^{-i\lambda L}\widehat{u}(i\lambda,t) - e^{i\lambda L}\widehat{u}(-i\lambda,t)\right]\right\}, \tag{2.20}$$

$$\widetilde{h}_1 = \frac{1}{e^{i\lambda L}-e^{-i\lambda L}}\left\{G(\lambda,t) - G(-\lambda,t) + e^{\lambda^2 t}\left[\widehat{u}(i\lambda,t) - \widehat{u}(-i\lambda,t)\right]\right\}. \tag{2.21}$$

Next, we substitute the obtained relations for $\widetilde{g}_1$ and $\widetilde{h}_1$ into Eq. (2.16). Here the integrals containing $\widehat{u}(\pm i\lambda,t)$ turn to zero in the obtained relation. Let us show it.

Let $\operatorname{Im}\lambda \geq 0$. Then by L'Hôpital rule the expression

$$\frac{e^{-i\lambda L}\widehat{u}(i\lambda,t) - e^{i\lambda L}\widehat{u}(-i\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}$$

is equivalent to

$$-\widehat{u}(i\lambda,t) + e^{i\lambda L}\int_0^L e^{i\lambda(L-x)}u(x,t)dx$$

as $\lambda \to \infty$ in any direction in $\operatorname{Im}\lambda \geq 0$, which is bounded.

Similarly, in the case where $\operatorname{Im}\lambda \leq 0$, the expression

$$\frac{\widehat{u}(-i\lambda,t) - \widehat{u}(i\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}$$

is equivalent to the bounded expression

$$\int_0^L e^{-i\lambda(L-x)}u(x,t)dx - e^{-i\lambda L}\widehat{u}(i\lambda,t)$$

as $\lambda\to\infty$ in any direction in $\operatorname{Im}\lambda\le 0$.

Also note that the denominator of these expressions $e^{i\lambda L}-e^{-i\lambda L}$ has only real zeroes and, hence, does not have zeroes in $D^{\pm}$, but $\lambda=0$. But this singularity is removable, hence we assume that these expressions are bounded in D^+ and D^-, respectively. This completes the proof of our statement.

Thus, we obtain the solution to the problem in the form

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\widehat{u}_0(\lambda)d\lambda-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[i\lambda\widetilde{g}_0(\lambda^2,t)+\frac{e^{i\lambda L}G(\lambda,t)-e^{-i\lambda L}G(-\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}\right]d\lambda$$
$$-\frac{1}{2\pi}\int_{\partial D^-} e^{i\lambda x-\lambda^2 t}\left[i\lambda\widetilde{h}_0(\lambda^2,t)+\frac{G(\lambda,t)-G(-\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}\right]d\lambda. \quad (2.22)$$

2.3. Star graph with finite and semifinite edges. In this section we give our result for the star graph from [12]. Consider a metric graph consisting of n finite edges $b_1, b_2,\ldots, b_n$ and m semifinite edges $b_{n+1}, b_{n+2},\ldots, b_{n+m}$ having the common vertex O called the *graph vertex*. Define the coordinates of each edge with the following relations: $b_j\sim(0,\ L_j)$ for $j=\overline{1,n}$ and $B_r\sim(0,+\infty)$ for $r=\overline{n+1,n+m}$. Here the vertex of the graph corresponds to 0 at each edge of the graph (Fig. 2.6).

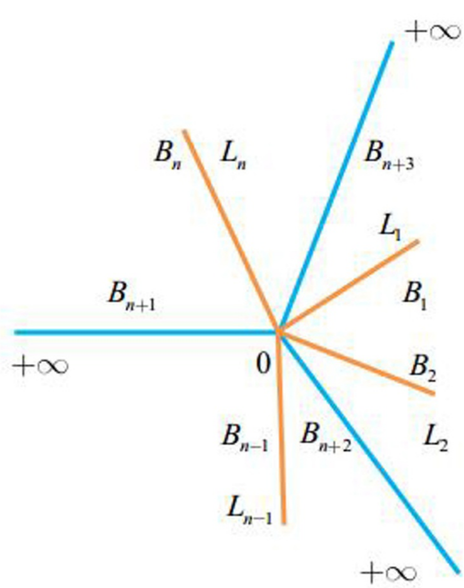

Fig. 2.6

We claim that in semifinite edges there hold the asymptotic conditions

$$\lim_{x\to\infty}u^{(r)}(x,t)=0,\quad 0<t\le T,\quad r=\overline{n+1,n+m}. \quad (2.23)$$

In this case the global relation for each solution is the same as in the cases considered above. With the boundary conditions, the initial condition and the conditions on the vertex from the global relations we obtain the system of $2n+m$ equations with respect to $2n+m$ boundary conditions. The main matrix of this system of equations has zeroes on the real axis, but $\lambda = 0$, which leads to the removable singularity in the integral relation and can be omitted. Next, analogically one can obtain the solution to the problem. The details are given in [12].

Theorem 2.1 (see [12]). *The solution of the considered initial-boundary value problem has the form*

$$
\begin{aligned}
u^{(j)}(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} & e^{ikx-wt}u_0^{(j)}(k)dk \\
&- \frac{1}{2\pi}\int\limits_{\partial D_R^-} e^{ikx-ikL_j-wt}\frac{\widehat{u}_0^{(j)}(k)-\widehat{u}_0^{(j)}(-k)-2ik\sigma^2\widetilde{g}_0(w,t)}{A_j}dk \\
&- \frac{1}{2\pi}\int\limits_{\partial D_R^+} e^{ikx-wt}\frac{e^{ikL_j}\widehat{u}_0^{(j)}(k)-e^{-ikL_j}\widehat{u}_0^{(j)}(-k)+2ik\sigma^2 h_0^{(j)}(w,t)}{A_j}dk, \quad j=\overline{1,n},
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
u^{(r)}(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{ikx-wt}\widehat{u}_0^{(r)}(k)dk - \frac{1}{2\pi}\int\limits_{\partial D^+} e^{ikx-wt}\widehat{u}_0^{(r)}(-k)dk& \\
- \frac{1}{\pi}\int\limits_{\partial D^+} e^{ikx-wt}ik\sigma^2\widetilde{g}_0(w,t)dk,\ (r=\overline{n+1,n+m}).&
\end{aligned}
\tag{2.25}
$$

Here

$$
\begin{aligned}
ik\sigma^2\widetilde{g}_0(w,t) = &\frac{1}{\sum\limits_{j=1}^{n}\delta_j^2\frac{B_j}{A_j}+\sum\limits_{r=n+1}^{n+m}\delta_r^2} \\
&\times\left[\sum_{j=1}^{n}\frac{\delta_j^2}{A_j}\left[e^{ikL_j}\widehat{u}_0^{(j)}(k)-e^{-ikL_j}\widehat{u}_0^{(j)}(-k)+2ik\sigma^2 h_0^{(j)}(w,t)\right]+\sum_{r=n+1}^{n+m}\delta_r^2\widehat{u}_0^{(r)}(k)\right], \\
&A_j = e^{ikL_j}-e^{-ikL_j},\quad B_j = e^{ikL_j}+e^{-ikL_j}\quad (j=\overline{1,n}).
\end{aligned}
$$

An analogical result in the case of nonstationary Schrödinger equation was obtained in [13].

3. Main Results

Let us now move on to the main results of our paper. In this section we apply the Fokas method to solve the initial-boundary value problem (2.1)–(2.4) in the case of more general graphs. We begin with the graph consisting of triangular part which has one connected edge at each vertex of this triangle (see Fig. 3.1).

3.1. The graph in the form of the triangle with edges connected to each vertex. In this case the number of edges $n = 6$, and the number of vertices $m = 6$. Here we give the method of the construction of the solution in details.

Let us start with the standard step in the Fokas method (see [6–8, 15–17]). We rewrite the equation in the form

$$
\left(e^{-ikx+wt}u^{(j)}(x,t)\right)_t = \left(e^{-ikx+wt}\left(u_x^{(j)}(x,t)+iku^{(j)}(x,t)\right)\right)_x, \tag{3.1}
$$

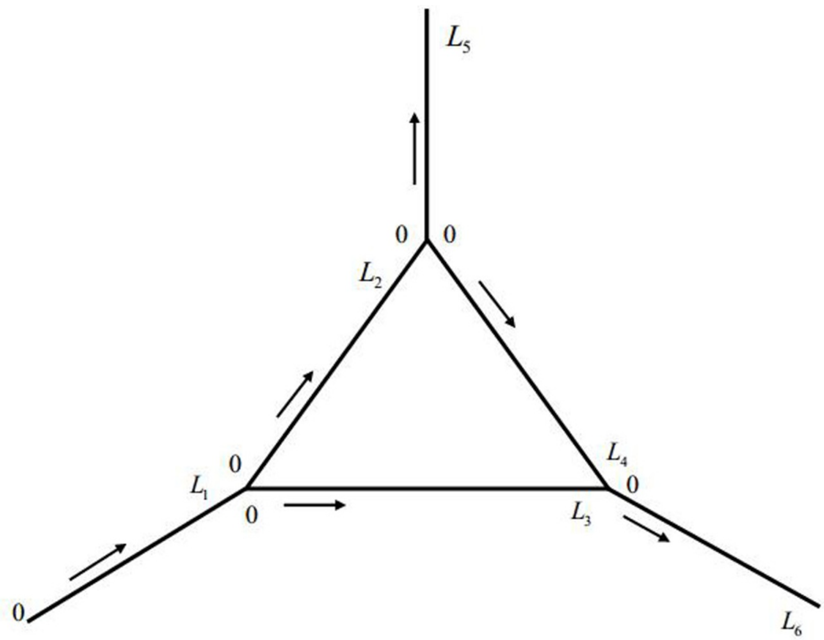

Fig. 3.1. The metric graph Γ_1

where $w(k) = k^2$, $j = 1, 2, \dots, 6$.

Integrating by the domain $(0,\ L_j) \times (0,\ t)$, $j = 1, 2, \dots, 6$, and applying Green's formula, we obtain

$$\int_0^{L_j} e^{-ikx+wt} u^{(j)}(x,t)dx - \int_0^{L_j} e^{-ikx} u^{(j)}(x,0)dx$$
$$= \int_0^t e^{-ikL_j+ws} \left(u_x^{(j)}(L_j,s) + iku^{(j)}(L_j,s)\right) ds - \int_0^t e^{ws} \left(u_x^{(j)}(0,s) + iku^{(j)}(0,s)\right) ds, \quad (3.2)$$

where $k \in \mathbb{C}$, $j = 1, 2, \dots, 6$.

Let us introduce the following notation for $k \in \mathbb{C}$:

$$\widehat{u}^{(j)}(k,t) = \int_0^{L_j} e^{-ikx} u^{(j)}(x,t)dx; \widehat{u}_0^{(j)}(k) = \int_0^{L_j} e^{-ikx} u_0^{(j)}(x)dx;$$

$$\widehat{h}_1^{(j)}(w,t) = \int_0^t e^{ws} u_x^{(j)}(L_j,s)ds; \widehat{h}_0^{(j)}(w,t) = \int_0^t e^{ws} u^{(j)}(L_j,s)ds;$$

$$\widehat{g}_1^{(j)}(w,t) = \int_0^t e^{ws} u_x^{(j)}(0,s)ds; \widehat{g}_0^{(j)}(w,t) = \int_0^t e^{ws} u^{(j)}(0,s)ds, \quad j = 1, 2, \dots, 6.$$

By this notation we can write Eq. (3.2), i.e., the global relation, in the form

$$e^{wt}\widehat{u}^{(j)}(k,t) - \widehat{u}_0^{(j)}(k) = e^{-ikL_j}(\widehat{h}_1^{(j)}(w,t) + ik\widehat{h}_0^{(j)}(w,t)) - \widehat{g}_0^{(j)}(w,t) - ik\widehat{g}_0^{(j)}(w,t), \quad j = 1, 2, \dots, 6, \ (3.3)$$

where $\{k \in \mathbb{C} : \operatorname{Im} k < 0\}$.

Susbtituting k with $-k$, from Eq. (3.3) we obtain

$$e^{wt}\widehat{u}^{(j)}(-k,t)-\widehat{u}_0^{(j)}(-k) = e^{ikL_j}\sigma^2(\widehat{h}_1^{(j)}(w,t)-ik\widehat{h}_0^{(j)}(w,t))-\sigma^2(\widehat{g}_j(w,t)-ik\widehat{g}_0(w,t)), \quad j=\overline{1,6}, \tag{3.4}$$

where $\{k \in \mathbb{C} : \operatorname{Im} k > 0\}$.

We apply the inverse Fourier transformation in the global relation (3.3):

$$u^{(j)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\widehat{u}_0^{(j)}(k)dk + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-ikL_j-wt}\left(\widehat{h}_1^{(j)}(w,t)+ik\widehat{h}_0^{(j)}(w,t)\right)dk$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\left(\widehat{g}_1^{(j)}(w,t)+ik\widehat{g}_0^{(j)}(w,t)\right)dk, \quad j=1,2,\ldots,6. \tag{3.5}$$

Let $\mathbb{C}^+ = \{k \in \mathbb{C} : \operatorname{Im} k > 0\}$, $\mathbb{C}^- = \{k \in \mathbb{C} : \operatorname{Im} k < 0\}$, $D = \left\{k \in \mathbb{C}^{\pm} : \operatorname{Re} k^2 < 0\right\} = D^+ \cup D^-$. The domain D is shown on Fig. 2.5.

The integrand in the second integral in Eq. (3.5) is an integer function and decreases as $k \to \infty$ in $k \in \mathbb{C}^-\backslash D^-$. The integrand in the third integral is an integer function and decreases as $k \to \infty$ in $k \in \mathbb{C}^+\backslash D^+$. Using the holomorph property of the integrands and the Jordan lemma one can substitute the integration contour in the second integral with $-\int\limits_{\partial D^-}$, and in the second integral with $-\int\limits_{\partial D^+}$ (see [7]):

$$u^{(j)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-wt}\widehat{u}_0^{(j)}(k)dk - \frac{1}{2\pi}\int_{\partial D^-} e^{ikx-ikL_j-wt}\left(\widehat{h}_1^{(j)}(w,t)+ik\widehat{h}_0^{(j)}(w,t)\right)dk$$
$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-wt}\left(\widehat{g}_1^{(j)}(w,t)+ik\widehat{g}_0^{(j)}(w,t)\right)dk, \quad j=1,2,\ldots,6. \tag{3.6}$$

Let us find the unknown functions in this integral representation of the solution. From the continuity conditions and the flow preservation equation (2.3) we obtain

$$\widehat{h}_0^{(1)}(w,t) = \widehat{g}_0^{(2)}(w,t) = \widehat{g}_0^{(3)}(w,t); \quad \widehat{h}_1^{(1)}(w,t) = \widehat{g}_1^{(2)}(w,t) + \widehat{g}_1^{(3)}(w,t);$$
$$\widehat{h}_0^{(2)}(w,t) = \widehat{g}_0^{(4)}(w,t) = \widehat{g}_0^{(5)}(w,t); \quad \widehat{h}_1^{(2)}(w,t) = \widehat{g}_1^{(4)}(w,t) + \widehat{g}_1^{(5)}(w,t);$$
$$\widehat{h}_0^{(3)}(w,t) = \widehat{h}_0^{(4)}(w,t) = \widehat{g}_0^{(6)}(w,t); \quad \widehat{h}_1^{(3)}(w,t) + \widehat{h}_1^{(4)}(w,t) = \widehat{g}_1^{(6)}(w,t).$$

Using the global relations from here we obtain the following system of equations:

$$e^{wt}\left(A_2A_3\left(\widehat{u}^{(1)}(k,t)-\widehat{u}^{(1)}(-k,t)\right) - A_1A_3\left(e^{ikL_2}\widehat{u}^{(2)}(k,t)-e^{-ikL_2}\widehat{u}^{(2)}(-k,t)\right)\right)$$
$$-e^{wt}A_1A_2\left(e^{ikL_3}\widehat{u}^{(3)}(k,t)-e^{-ikL_3}\widehat{u}^{(3)}(-k,t)\right) - A_1A_3\left(\widehat{u}_0^{(1)}(k)-\widehat{u}_0^{(1)}(-k)\right)$$
$$+A_1A_3\left(e^{ikL_2}\widehat{u}_0^{(2)}(k)-e^{-ikL_2}\widehat{u}_0^{(2)}(-k)\right) + A_1A_2\left(e^{ikL_3}\widehat{u}_0^{(3)}(k)-e^{-ikL_3}\widehat{u}_0^{(3)}(-k)\right)$$
$$= ikC_{123}\widehat{h}_0^{(1)}(w,t) - 2ikA_1A_3\widehat{h}_0^{(2)}(w,t) - 2ikA_1A_2\widehat{h}_0^{(3)}(w,t) - 2ikA_2A_3\widehat{g}_0^{(1)}(w,t); \tag{3.7}$$

$$e^{wt}\left(A_4A_5\left(\widehat{u}^{(2)}(k,t)-\widehat{u}^{(2)}(-k,t)\right) - A_2A_5\left(e^{ikL_4}\widehat{u}^{(4)}(k,t)-e^{-ikL_4}\widehat{u}^{(4)}(-k,t)\right)\right)$$
$$-e^{wt}A_2A_4\left(e^{ikL_5}\widehat{u}^{(5)}(k,t)-e^{-ikL_5}\widehat{u}^{(5)}(-k,t)\right) - A_4A_5\left(\widehat{u}_0^{(2)}(k)-\widehat{u}_0^{(2)}(-k)\right)$$

$$+ A_2A_5\left(e^{ikL_4}\widehat{u}_0^{(4)}(k) - e^{-ikL_4}\widehat{u}_0^{(4)}(-k)\right) + A_2A_4\left(e^{ikL_5}\widehat{u}_0^{(5)}(k) - e^{-ikL_5}\widehat{u}_0^{(5)}(-k)\right)$$
$$= -2ikA_4A_5\widehat{h}_0^{(1)}(w,t) + ikC_{245}\widehat{h}_0^{(2)}(w,t) - 2ikA_2A_5\widehat{h}_0^{(3)}(w,t) - 2ikA_2A_4\widehat{h}_0^{(5)}(w,t); \quad (3.8)$$

$$e^{wt}\left(A_4A_6\left(\widehat{u}^{(3)}(k,t) - \widehat{u}^{(3)}(-k,t)\right) + A_3A_6\left(\widehat{u}^{(4)}(k,t) - \widehat{u}^{(4)}(-k,t)\right)\right)$$
$$- e^{wt}A_3A_4\left(e^{ikL_6}\widehat{u}^{(6)}(k,t) - e^{-ikL_6}\widehat{u}^{(6)}(-k,t)\right) - A_4A_6\left(\widehat{u}_0^{(3)}(k) - \widehat{u}_0^{(3)}(-k)\right)$$
$$- A_3A_6\left(\widehat{u}_0^{(4)}(k) - \widehat{u}_0^{(4)}(-k)\right) + A_3A_4\left(e^{ikL_6}\widehat{u}_0^{(6)}(k) - e^{-ikL_6}\widehat{u}_0^{(6)}(-k)\right)$$
$$= -2ikA_4A_6\widehat{h}_0^{(1)}(w,t) - 2ikA_3A_6\widehat{h}_0^{(2)}(w,t) - 2ikA_3A_4\widehat{h}_0^{(6)}(w,t) + ikC_{346}\widehat{h}_0^{(3)}(w,t). \quad (3.9)$$

Here $\widehat{h}_0^{(j)}(w,t)$, $j = 1,2,3$ are unknown functions, $C_{ijk} = B_iA_jA_k + A_iB_jA_k + A_iA_jB_k$.

We rewrite the system of Eqs. (3.7)–(3.9) in the form

$$M \cdot X(w,t) = N_1(k)\cdot\widehat{U}_0(k) + N_2(k)\cdot\widehat{U}_0(-k)) - e^{wt}\cdot(N_1(k)\cdot\widehat{U}(k,t) + N_2(k)\cdot\widehat{U}(-k,t)), \quad (3.10)$$

where

$$X(w,t) = \left(\widehat{h}_0^{(1)}(w,t), \widehat{h}_0^{(2)}(w,t), \widehat{h}_0^{(3)}(w,t)\right)^T.$$

In order to prove the solvability of the latter system of equations we require two following lemmas.

Lemma 3.1. *Problem* (2.1)–(2.4) *has no more than one regular solution.*

Proof. Consider problem (2.1)–(2.4) with homogeneous boundary conditions. Multiplying Eq. (2.1) by $u^{(j)}(x,t)$ and integrating from 0 to L_j for each $j = \overline{1,n}$, adding the obtained equalities and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\int_0^{L_j}(u^{(j)}(x,t))^2dx + \sum_{j=1}^{n}\int_0^{L_j}(u_x^{(j)}(x,t))^2dx = 0.$$

Next, integrating this equation by t, we obtain

$$\sum_{j=1}^{n}\int_0^{L_j}(u^{(j)}(x,t))^2dx + 2\sum_{j=1}^{n}\int_0^{t}\int_0^{L_j}(u_x^{(j)}(x,s))^2dxds = \sum_{j=1}^{n}\int_0^{L_j}(u_0^{(j)}(x))^2dx.$$

From here it follows that if $u_0^{(j)}(x) \equiv 0$, then $u^{(j)}(x,t) = 0$ as $x \in b_j$, $t > 0$. This completes the proof of the lemma. □

Lemma 3.2. *The determinant of the matrix*

$$M(k) = \begin{pmatrix} C_{123} & -2A_1A_3 & -2A_1A_2 \\ -2A_4A_5 & C_{245} & -2A_2A_5 \\ -2A_4A_6 & -2A_3A_6 & C_{346} \end{pmatrix}$$

is not zero as $\operatorname{Im} k \neq 0$.

Proof. Let $\psi_j(x)$ be the solution of the equation $-\dfrac{d^2}{dx^2}\psi(x) = \lambda\psi(x)$ on the graph Γ with homogeneous boundary conditions (2.4) and gluing conditions (Kirchhoff conditions) given above. We set $\lambda = k^2$. It is known that in that case the graph is called the *quantum graph* [5]. The spectral problems on the quantum graphs are studied in [3, 9, 10].

Consider problem (2.1)–(2.4) with homogeneous boundary conditions and the initial condition $u^{(j)}(x,0) = \psi_j(x)$, $x \in \bar{b}_j$, $j = \overline{1,n}$. It is clear that this problem has a unique solution of the form $u^{(j)}(x,t) = e^{-k^2t}\psi_j(x)$. Substituting this expression into system (3.7)–(3.9), we obtain the system of equations $M(k)\phi = 0$, where $\phi = (\psi_1(L_1), \psi_2(L_2)), \psi_3(L_3))$.

Suppose that $\det M(k) = 0$ for some k, the imaginary part of which is nonzero. Then the system of equations $M(k)\phi = 0$ has nonzero solutions. Next, from the global relations (3.3) one can find the derivatives $\phi' = (\psi_1'(L_1), \psi_2'(L_2)), \psi_3'(L_3))$.

Since the second derivative is invariant with respect to the substitution of x with $L_j - x$ on each separate edge b_j, $j = \overline{1, n}$, we obtain the Cauchy problem for the equation $\psi_j''(x) = k^2\psi(x)$ with the initial data obtained above. Since at least one of the initial values differs from zero, at least one of these Cauchy problems has a nonzero solution. From the global relation (3.3) and the system of Eqs. (3.7)–(3.9) it follows that the obtained solutions satisfy the boundary conditions (2.4) and gluing conditions (Kirchhoff conditions) (2.3) at the vertices of the graph.

Thus, we obtained the eigenfunctions for the Schrödinger operator $-\dfrac{d^2}{dx^2}$ on the metric graph Γ corresponding to the eigenvalue $\lambda = k^2$. By our proposition this eigenvalue is either negative or a complex number with nonzero imaginary part. But is contradicts to the positive definiteness and self-adjointness of the Schrödinger operator on the metric graph Γ (see [3, 9, 10]).

This completes the proof of the lemma. $\square$

Note that as $k \to 0$ both parts of the system of equations have the order k^3, and then the solution of the system of Eqs. (3.7)–(3.9) has a removable singularity as $k = 0$.

Consider the integrals $\int\limits_{\partial D^+} e^{ikx}\widehat{u}(\pm k, t)dk$ and $\int\limits_{\partial D^-} e^{-ik(L_j - x)}\widehat{u}(\pm k, t)dk$. It is known that the integrands in these integrals are holomorph functions in D^- and D^+, respectively (by the regularity of the solution of the problem), and tend to zero exponentially in these domains as $\lambda \to \infty$. Hence, by the Cauchy theorem and the Jordan lemma these integrals are equal to zero. Thus, we can omit the third and fourth terms in the right-hand part of Eq. (3.10) consisting of an unknown solution, since they vanish under the substitution into the integral representation of the solution.

In the conclusion of this subsection we state the obtained result in the form of the following theorem.

Theorem 3.1. *The solution of the initial-boundary value problem on* Γ_1 *has the form*

$$u^{(j)}(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{ikx - wt}\widehat{u}_0^{(j)}(k)dk - \frac{1}{2\pi}\int\limits_{\partial D^-} e^{ikx - ikL_j - wt}\left(\tilde{h}_1^{(j)}(w,t) + ik\tilde{h}_0^{(j)}(w,t)\right)dk$$
$$- \frac{1}{2\pi}\int\limits_{\partial D^+} e^{ikx - wt}\left(\tilde{g}_1^{(j)}(w,t) + ik\widehat{g}_0^{(j)}(w,t)\right)dk, \quad j = 1, 2, \ldots, 6, \quad (3.11)$$

where $G^{(j)}(k,t) = \widehat{u}_0^{(j)}(k) + (e^{-ikL_j} - 1)ik\widehat{h}_0^{(j)}(w,t)$, $j = 1, 2, \ldots, 6$, $\widehat{g}_0^{(1)}(w,t)$, $\widehat{h}_0^{(5)}(w,t)$, $\widehat{h}_0^{(6)}(w,t)$ *are known functions, which are expressed via the boundary data as in Eq.* (2.4),

$$\tilde{h}_1^{(j)}(w,t) = \frac{1}{A_j}\left(G^{(j)}(k,t) - G^{(j)}(-k,t)\right), \quad (3.12)$$

$$\tilde{g}_1^{(j)}(w,t) = \frac{1}{A_j}\left(e^{ikL_j}G^{(j)}(k,t) - e^{-ikL_j}G^{(j)}(-k,t)\right), \quad (3.13)$$

$A_j = e^{ikL_j} - e^{-ikL_j}$, $B_j = e^{ikL_j} + e^{-ikL_j}$, $j = 1, 2, \ldots, 6$, $\left(\tilde{h}_0^{(1)}(w,t), \tilde{h}_0^{(2)}(w,t), \tilde{h}_0^{(3)}(w,t)\right)^T = (M(k))^{-1}$ $\times(N_1(k)\cdot\widehat{U}_0(k) + N_2(k)\cdot\widehat{U}_0(-k))$.

3.2. Tree graph. Consider an initial-boundary value problem in the case of the tree graph Γ_2. In this case $n = 7$ and $m = 8$.

The solution to the problem in this case is constructed analogically, thus we omit the details of its construction. In this case the main system of equations for finding the value of the solution at branching points has the form:

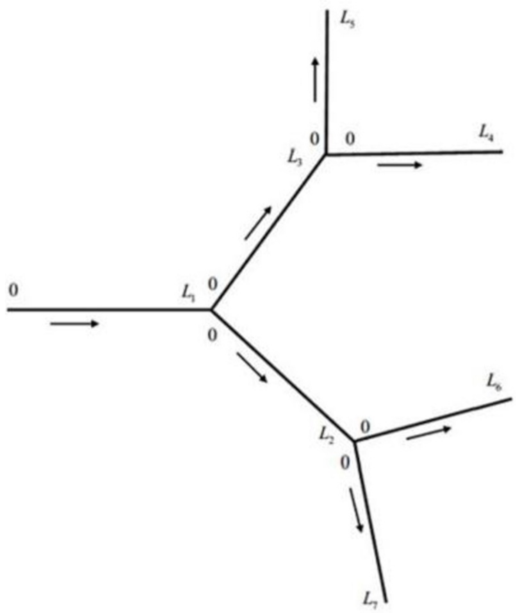

FIG. 3.2.1. The metric graph Γ_2

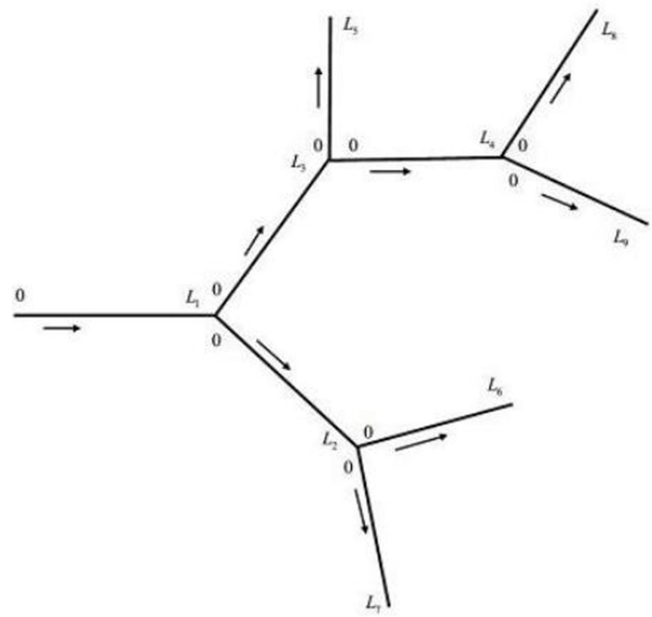

FIG. 3.2.2. The metric graph Γ_3

$$\begin{aligned}
&e^{wt}\left(A_2A_3\left(\widehat{u}^{(1)}(k,t)-\widehat{u}^{(1)}(-k,t)\right)-A_1A_3\left(e^{ikL_2}\widehat{u}^{(2)}(k,t)-e^{-ikL_2}\widehat{u}^{(2)}(-k,t)\right)\right)\\
&\qquad -e^{wt}A_1A_2\left(e^{ikL_3}\widehat{u}^{(3)}(k,t)-e^{-ikL_3}\widehat{u}^{(3)}(-k,t)\right)-A_1A_3\left(\widehat{u}_0^{(1)}(k)-\widehat{u}_0^{(1)}(-k)\right)\\
&\qquad +A_1A_3\left(e^{ikL_2}\widehat{u}_0^{(2)}(k)-e^{-ikL_2}\widehat{u}_0^{(2)}(-k)\right)+A_1A_2\left(e^{ikL_3}\widehat{u}_0^{(3)}(k)-e^{-ikL_3}\widehat{u}_0^{(3)}(-k)\right)\\
&\qquad =ikC_{123}\widehat{h}_0^{(1)}(w,t)-2ikA_1A_3\widehat{h}_0^{(2)}(w,t)-2ikA_1A_2\widehat{h}_0^{(3)}(w,t)-2ikA_2A_3\widehat{g}_0^{(1)}(w,t);
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&e^{wt}\left(A_4A_5\left(\widehat{u}^{(3)}(k,t)-\widehat{u}^{(3)}(-k,t)\right)-A_3A_5\left(e^{ikL_4}\widehat{u}^{(4)}(k,t)-e^{-ikL_4}\widehat{u}^{(4)}(-k,t)\right)\right)\\
&\qquad -e^{wt}A_3A_4\left(e^{ikL_5}\widehat{u}^{(5)}(k,t)-e^{-ikL_5}\widehat{u}^{(5)}(-k,t)\right)-A_4A_5\left(\widehat{u}_0^{(3)}(k)-\widehat{u}_0^{(3)}(-k)\right)\\
&\qquad +A_3A_5\left(e^{ikL_4}\widehat{u}_0^{(4)}(k)-e^{-ikL_4}\widehat{u}_0^{(4)}(-k)\right)+A_3A_4\left(e^{ikL_5}\widehat{u}_0^{(5)}(k)-e^{-ikL_5}\widehat{u}_0^{(5)}(-k)\right)\\
&\qquad =ikC_{345}\widehat{h}_0^{(3)}(w,t)-2ikA_4A_5\widehat{h}_0^{(1)}(w,t)-2ikA_3A_5\widehat{h}_0^{(4)}(w,t)-2ikA_3A_4\widehat{h}_0^{(5)}(w,t);
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&e^{wt}\left(A_6A_7\left(\widehat{u}^{(2)}(k,t)-\widehat{u}^{(2)}(-k,t)\right)-A_2A_7\left(e^{ikL_6}\widehat{u}^{(6)}(k,t)-e^{-ikL_6}\widehat{u}^{(6)}(-k,t)\right)\right)\\
&\qquad -e^{wt}A_2A_6\left(e^{ikL_7}\widehat{u}^{(7)}(k,t)-e^{-ikL_7}\widehat{u}^{(7)}(-k,t)\right)-A_6A_7\left(\widehat{u}_0^{(2)}(k)-\widehat{u}_0^{(2)}(-k)\right)\\
&\qquad +A_2A_7\left(e^{ikL_6}\widehat{u}_0^{(6)}(k)-e^{-ikL_6}\widehat{u}_0^{(6)}(-k)\right)+A_2A_6\left(e^{ikL_7}\widehat{u}_0^{(7)}(k)-e^{-ikL_7}\widehat{u}_0^{(7)}(-k)\right)\\
&\qquad =ikC_{267}\widehat{h}_0^{(2)}(w,t)-2ikA_6A_7\widehat{h}_0^{(1)}(w,t)-2ikA_2A_7\widehat{h}_0^{(6)}(w,t)-2ikA_2A_6\widehat{h}_0^{(7)}(w,t).
\end{aligned}\tag{3.16}$$

The existence of solutions of this system of equations is proved analogically to that above. After we find these unknowns, the problem is reduced to the solution of the initial-boundary value problem on the finite interval given above.

For the illustration we give another case of the tree graph Γ_3 (see Fig. 3.2.2), where the number of edges $n = 9$ and the number of vertices $m = 10$.

In this case the number of the internal vertices in equal to four, since after the addition of additional edges one of the boundary vertices of the graph Γ_2 becomes an internal vertex. Hence, in the system of Eqs. (3.14)–(3.16) in order to find the unknown values of the solutions at the internal vertices (at the branching points) of the graph we need to add another equation:

$$\begin{aligned} e^{wt}\Big(A_8A_9\Big(\widehat{u}^{(4)}(k,t)-\widehat{u}^{(4)}(-k,t)\Big)-A_4A_9\Big(e^{ikL_8}\widehat{u}^{(8)}(k,t)-e^{-ikL_8}\widehat{u}^{(8)}(-k,t)\Big)\Big) \\ -e^{wt}A_4A_8\Big(e^{ikL_9}\widehat{u}^{(9)}(k,t)-e^{-ikL_9}\widehat{u}^{(9)}(-k,t)\Big)-A_8A_9\Big(\widehat{u}_0^{(4)}(k)-\widehat{u}_0^{(4)}(-k)\Big) \\ +A_4A_9\Big(e^{ikL_8}\widehat{u}_0^{(8)}(k)-e^{-ikL_8}\widehat{u}_0^{(8)}(-k)\Big)+A_4A_8\Big(e^{ikL_9}\widehat{u}_0^{(9)}(k)-e^{-ikL_9}\widehat{u}_0^{(9)}(-k)\Big) \\ =ikC_{489}\widehat{h}_0^{(4)}(w,t)-2ikA_8A_9\widehat{h}_0^{(3)}(w,t)-2ikA_4A_9\widehat{h}_0^{(8)}(w,t)-2ikA_4A_8\widehat{h}_0^{(9)}(w,t). \quad (3.17) \end{aligned}$$

4. Conclusion and Results

In this paper we gave the method of the construction of solutions of initial-boundary value problems on some metric graphs, such as star graph; triangular graph with additional edges connected to each vertex; tree graph. We chose these graphs to show that in the case of a graph with cycles and graphs without cycles the method works almost the same. In general, this method is applicable to the solution of the initial-boundary value problems in arbitrary metric graphs.

While solving the problem we use the so-called *global relation* and the relation obtained from it by the substitution of the complex parameter with the opposite. These two relations are equivalent to the mapping of the Dirichlet conditions onto the Neumann conditions at vertices. With these relations the considered problem is reduced to the system of algebraic equations with respect to the unknown values of the solution at the branching points of the graph.

We give the solution in the form of the contour integrals of unknown functions. With that the contours are chosen in such way that the integrands decrease exponentially an infinity on this contour. This property provides good convergence of the integrals, which is very important, for instance, for the numerical computation of the solution (see [11]).

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