Feshbach-Schur projection

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Abstract

The orthogonalizing pseudopotential (OPP) method is widely used to exclude Pauli-forbidden states in microscopic cluster and few-body models. Traditionally, the OPP introduces a large strength parameter λ whose numerical finiteness leads to residual dependence of observables on λ. We present a Feshbach-Schur projection formalism (FSP) which yields observables independent of λ and provides practical routes for constructing Pauli-corrected potentials in few-body theory.

1 Introduction

A central challenge in cluster models is the exclusion of Pauli-forbidden components arising from internal antisymmetrization. The *Orthogonality Condition Model* (OCM) of Saito introduced explicit orthogonality constraints to forbidden relative-motion states in cluster channels [1, 2]. In parallel, Kukulin and collaborators formulated the *Orthogonalizing Pseudopotential* (OPP) approach to enforce orthogonality numerically [3, 4], adding a rank-n_f separable repulsion

$$V_{\mathrm{OPP}} = \lambda \sum_f |\phi_f\rangle\langle\phi_f|, \tag{1}$$

with λ a large positive constant. In the formal limit $\lambda \to \infty$, the forbidden components are completely projected out; however, in practice λ remains finite, and calculated observables may show a small but noticeable dependence on λ. OPP variants and their projection-operator interpretation have been used widely in nuclear cluster physics and are conceptually connected to Feshbach projection and Schur complements [5, 6].

General reviews of related techniques, including orthogonalizing pseudopotentials, appear in broader few-body contexts as well [7], though primarily in atomic systems. For historical overviews of microscopic cluster models and OCM/OPP usage, see Fujiwara *et al.* [8].

Deep $\alpha\alpha$ potentials with embedded Pauli-forbidden S- and D-wave states have long served as a testing ground for OPP/OCM methods, with orthogonality removing spurious bound states while preserving realistic phase shifts and resonances. The αn system, typically containing one S-wave forbidden state, is another standard benchmark. Many studies adopt large-but-finite λ and test the stability of observables against λ-variation; this regime also allows nonlocal projectors or direct orthogonalization to be implemented within variational or integral-equation frameworks [8].

A projection-based formulation of the OCM was introduced in Refs. [9,10], where the forbidden α–α components are removed by defining the projected interaction

$$V_{\rm proj} = QVQ, \qquad Q = 1 - \sum_f |\phi_f\rangle\langle\phi_f|.$$

The kinetic term is unchanged, so the effective Hamiltonian has the form $H_{\rm proj} = H_0 + QVQ$. This yields a shallow phase-equivalent but nonlocal potential without introducing a large pseudopotential strength λ. Because the projection acts in a single relative coordinate, the construction is feasible in the two-body α–α problem. Extending it to three-body systems is nontrivial, since pairwise projectors do not commute across different Jacobi partitions and generate fully nonlocal three-body kernels. Consequently, most few-body calculations continue to use OPP with finite λ, or Pauli-safe basis functions, rather than QVQ at the Hamiltonian level.

Three-α calculations (e.g., ^{12}C) explicitly require consistent removal of pairwise Pauli-forbidden components across Jacobi partitions. Recent works implement three distinct strategies: (i) OPP with large λ in variational Gaussian bases, monitoring convergence and the appearance of "almost forbidden states" (very small-norm components); (ii) direct (exact) orthogonalization to forbidden subspaces without tuning λ; and (iii) complete three-body projectors demonstrating formal equivalence of projector kernels in few-body space to the constraints used operationally in OPP.

As a case study, Tursunov *et al.* analyze the 0^+ and 2^+ states of ^{12}C in a 3α model with deep $\alpha\alpha$ potentials, comparing OPP with direct projector methods and discussing possible signatures of quantum phase–transition behavior [11, 12]. Experience with OPP shows that very large λ can lead to stiffness and slow convergence, motivating projector/basis alternatives.

In nucleon–nucleus coupled-channel problems, the *Multi-Channel Algebraic Scattering* (MCAS) method systematically removes Pauli-forbidden components by OPP-like orthogonalization among channel states. MCAS studies show how failure to enforce Pauli blocking leads to spurious states and demonstrate OPP-based cures within Lippmann–Schwinger frameworks [13, 14]. This illustrates a community where λ-regulated orthogonalization is routine and practically effective.

The OPP term has a natural rank-n_f separable structure. In momentum space, one works with $\tau(E) = [\lambda^{-1} - \mathcal{D}(E)]^{-1}$ and can incorporate forbidden-state form factors in an enlarged separable basis [15, 16]. Separable formulations expose how the $\lambda \to \infty$ limit maps to Schur complements/Feshbach projections on the resolvent and motivate λ-eliminated kernels acting solely in the Pauli-allowed subspace—useful in Faddeev/AGS calculations. While classic papers enforced Pauli blocking via large λ, recent work emphasizes explicit projector constructions or basis sets satisfying orthogonality by design.

Several recent studies aim to avoid large λ entirely by building Pauli-allowed spaces directly or by improving projector technology:

- correlated-Gaussian multi-α OCM bases [17, 18];
- basis states satisfying the Pauli principle *a priori* in OCM [19];
- Nilsson-based PF states in deformed halo systems [20].

These developments reinforce the trend toward *operator/basis* approaches that reduce or eliminate reliance on very large λ.

Experience across 2- and 3-cluster problems indicates:

1. Large-but-finite λ generally enforces orthogonality, but observables may retain residual λ-dependence, especially when forbidden states are approximate.

2. "Almost forbidden states" (AFS)—small-norm components under the projector—can slow convergence and cause numerical stiffness when λ is huge. Working directly in the allowed space alleviates this.

3. In coupled-channel (MCAS) applications, OPP interacts with channel truncations, making explicit projectors or Pauli-safe channel sets attractive.

The *Resonating Group Method* (RGM) provides an alternative and, in principle, *exact* treatment of the Pauli principle at the microscopic level. Starting from antisymmetrized cluster wave functions, RGM yields integro-differential equations with fully nonlocal kernels; Pauli exclusion is built in explicitly. In that sense, RGM performs the Pauli projection at the microscopic level and requires no pseudopotential. Comprehensive accounts are given in Refs. [8, 21].

By contrast, the OPP method begins with an effective two-body or cluster potential—usually local or separable—and *then* enforces Pauli exclusion via the rank-n_f operator $\lambda \sum_f |\phi_f\rangle\langle\phi_f|$. Formally, the auxiliary repulsion acts as a Feshbach-like projector that suppresses the same forbidden subspace that the RGM removes automatically, but with very different practical advantages.

In modern cluster physics the two approaches are often viewed as complementary. RGM provides a microscopic reference; OPP offers a computationally lightweight projection compatible with local/separable potentials and momentum-space few-body solvers. Recent Pauli-safe bases (Moriya, Horiuchi; Myo *et al.*) effectively combine the strengths of both: microscopic antisymmetrizatirXivon implemented at the basis level, without large λ. In this context, analytic λ-eliminated OPP formulations can be interpreted as the operator limit of these projectors, clarifying the link between phenomenological OPP and microscopic Pauli exclusion.

Thus, the OPP method is now a mature tool in cluster calculations, used in 2- and 3-body nuclear systems and in coupled-channel scattering. The literature supports both (i) pragmatic finite-λ implementations and (ii) projector/basis approaches that avoid tuning λ. Recent developments strengthen the latter trend by constructing Pauli-allowed subspaces directly. This review consolidates the relevant citations to enable careful comparisons, including contexts where explicit operator reformulations are advantageous.

To our knowledge, the explicit closed-form T-matrix for general separable potentials, expressed via a Schur complement, has not been widely documented. Building on this foundation, the present work provides (i) a unified operator formulation that makes the Schur-complement structure transparent for general separable potentials, and (ii) a configuration-space formulation based on direct projection of the Hamiltonian, which clarifies how OPP generates a nonlocal subtraction kernel acting only within the Pauli-allowed subspace.

2 Feshbach-Schur projection

Let a two-body potential be given in separable form,

$$V = \sum_{i,j} |\chi_i\rangle \lambda_{ij} \langle\chi_j|, \tag{2}$$

where λ is the coupling matrix in the basis $\{|\chi_i\rangle\}$. Inclusion of the OPP term (1) modifies this interaction as

$$\tilde{V} = V + \lambda_0 \sum_f |\phi_f\rangle\langle\phi_f|, \tag{3}$$

where λ_0 denotes the OPP strength parameter. For the combined separable set $\{|\chi_i\rangle, |\phi_f\rangle\}$, the coupling matrix takes the block form

$$\tilde{\boldsymbol{\lambda}} = \begin{pmatrix} \boldsymbol{\lambda} & 0 \\ 0^T & \lambda_0 \mathbf{I} \end{pmatrix}. \tag{4}$$

The two-body T-matrix has the standard separable structure

$$T(E) = \sum_{i,j} |\chi_i\rangle \tau_{ij}(E) \langle\chi_j|, \qquad \boldsymbol{\tau}(E) = [\boldsymbol{\lambda}^{-1} - \boldsymbol{\mathcal{D}}(E)]^{-1}, \tag{5}$$

where $\mathcal{D}_{ij}(E) = \langle\chi_i|G_0(E)|\chi_j\rangle$.

When the OPP term is included, Eq. (5) generalizes to

$$\tilde{\boldsymbol{\tau}}(E) = [\tilde{\boldsymbol{\lambda}}^{-1} - \tilde{\boldsymbol{\mathcal{D}}}(E)]^{-1}, \tag{6}$$

where $\tilde{\boldsymbol{\mathcal{D}}}$ includes matrix elements between both allowed and forbidden form factors. In the limit $\lambda_0 \to \infty$, we obtain from (4) the exact matrix identity

$$\lim_{\lambda_0 \to \infty} \tilde{\boldsymbol{\lambda}}^{-1} = \begin{pmatrix} \boldsymbol{\lambda}^{-1} & 0 \\ 0^T & 0 \end{pmatrix}. \tag{7}$$

Substituting this into the expression for $\tilde{\boldsymbol{\tau}}$, the block inversion yields

$$\boldsymbol{\tau}^{(\mathrm{eff})}(E) = [\boldsymbol{\lambda}^{-1} - \boldsymbol{\mathcal{D}}_{aa}(E) - \boldsymbol{\mathcal{D}}_{af}(E)\, \boldsymbol{\mathcal{D}}_{ff}^{-1}(E)\, \boldsymbol{\mathcal{D}}_{fa}(E)]^{-1}, \tag{8}$$

where subscripts a and f refer to allowed and forbidden components, respectively. Equation (8) defines the exact $\lambda_0 \to \infty$ limit of the OPP-modified T-matrix without any explicit reference to λ_0. This expression represents the Feshbach-Schur projection formalism.

The components of this derivation are well-established; however, the explicit block-matrix presentation and analytic elimination of the OPP strength parameter in the context of general separable potentials provides a compact formulation that is not widely available in the literature.

3 Extension to configuration-space formulation

Although the OPP method is frequently implemented numerically in coordinate space with a large finite λ [14], we present here an explicit analytic derivation of the exact $\lambda \to \infty$ limit. This formulation clarifies the structure of the nonlocal subtraction kernel and its relationship to approximate forms commonly used in the literature.

The OPP method was originally formulated in configuration space, where the auxiliary operator

$$V_{\mathrm{OPP}}(\mathbf{r}, \mathbf{r}') = \lambda\, \phi_f(\mathbf{r})\, \phi_f(\mathbf{r}') \tag{9}$$

is added to the microscopic potential in the Schrödinger equation,

$$[H + V_{\mathrm{OPP}} - E]\, \Psi(\mathbf{r}) = 0.$$

In this representation, the limit $\lambda \to \infty$ is only implicit: the forbidden components of the wave function are suppressed numerically by taking λ very large. All existing applications in coordinate space therefore use large but finite λ values, verifying the convergence of the observables.

We consider the radial, S-wave OPP Schrödinger equation for a two-body cluster system (here α–n is a canonical example). For the α–n interaction, the microscopic $0s$–shell antisymmetrization implies a single forbidden S-state $\phi_f(r)$. Let the reduced radial wave function be

$$u(r) = r\Psi(r),$$

and let the normalized forbidden radial function be

$$\langle r|\phi_f\rangle \equiv \phi_f(r), \qquad \langle \phi_f|\phi_f\rangle = 1.$$

Define the rank-one projector onto the forbidden function

$$P_f = |\phi_f\rangle \langle \phi_f|, \qquad Q = 1 - P_f.$$

The free Hamiltonian (kinetic energy operator) is

$$H_0 = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2},$$

with the usual boundary conditions for reduced radial waves (regular at $r = 0$, appropriate asymptotics as $r \to \infty$). The local potential is $V(r)$ and $H = H_0 + V$.

The OPP-modified radial Schrödinger equation reads

$$\big[\, H + \lambda P_f - E \,\big]\, u \;=\; 0. \tag{10}$$

Our goal is to eliminate the forbidden amplitude exactly and obtain an equation for $u_a \equiv Qu$, then take the $\lambda \to \infty$ limit and present the exact subtraction kernel. We will also explain how the commonly quoted subtraction in terms of $V(r')\phi_f(r)\phi_f(r')$ arises as an approximation or special case.

For that aim we decompose the wave function into allowed and forbidden parts,

$$u \;=\; u_a + c\,\phi_f, \qquad u_a \equiv Qu, \quad c \equiv \langle \phi_f|u\rangle. \tag{11}$$

By construction $\langle \phi_f|u_a\rangle = 0$.

Inserting (11) into (10) one gets:

$$(H - E)(u_a + c\phi_f) + \lambda P_f(u_a + c\phi_f) = 0.$$

Using $P_f u_a = 0$ and $P_f\phi_f = \phi_f$, this reduces to

$$(H - E)u_a + c(H - E)\phi_f + \lambda c\phi_f = 0. \tag{12}$$

To obtain an exact expression for c we take the inner product of (12) with $\langle \phi_f|$

$$\langle \phi_f|(H - E)u_a\rangle + c\langle \phi_f|(H - E)\phi_f\rangle + \lambda c\langle \phi_f|\phi_f\rangle = 0.$$

Since $\langle \phi_f|\phi_f\rangle = 1$, we define

$$\varepsilon_f \equiv \langle \phi_f|H|\phi_f\rangle,$$

and rearranging we get:

$$c \;=\; -\,\frac{\langle \phi_f | (H-E) u_a \rangle}{\varepsilon_f - E + \lambda}. \tag{13}$$

Equation (13) is exact for any finite λ. It shows that $c \to 0$ as $\lambda \to \infty$ (as physically expected).

To eliminate c entirely we substitute (13) back into (12). This yields the *exact* projected equation acting on u_a:

$$\big[\, H - E \,\big] u_a - (H-E)\,|\phi_f\rangle\, \frac{\langle \phi_f | (H-E) u_a \rangle}{\varepsilon_f - E + \lambda} \;=\; 0. \tag{14}$$

Equation (14) is the Schur-complement form of the original equation and is algebraically exact for finite λ.

From (13) it is immediate that $c \to 0$ for $\lambda \to \infty$, hence the forbidden component vanishes and the allowed component satisfies

$$(H-E)u_a = 0,$$

together with the orthogonality constraint $\langle \phi_f | u_a \rangle = 0$.

However, to obtain an explicit *nonlocal* subtraction kernel that displays how forbidden-channel coupling is removed (and which can be implemented numerically without large λ), it is instructive to rearrange the exact identity in a different way.

Starting from the identity $Q(H+\lambda P_f - E)Q = 0$ and noting $QP_f = 0$, we have directly

$$Q(H-E)Q\,u = 0 \quad \Longrightarrow \quad Q(H-E)u_a = 0.$$

Writing $Q = 1 - P_f$ and acting on u_a,

$$\big(H-E\big)u_a - P_f(H-E)u_a = 0.$$

Expressed in coordinate/radial representation this is

$$(H-E)u_a(r) - \phi_f(r) \int_0^\infty dr'\, \phi_f(r')\,(H-E)u_a(r') = 0. \tag{15}$$

This is exact and contains a rank-one nonlocal subtraction operator built from $\phi_f(r)$ and $(H-E)u_a$.

Because $H = H_0 + V$, and using self-adjointness of H_0 with the chosen boundary conditions, we may integrate by parts to move H_0 from u_a onto ϕ_f inside the integral (one must check boundary terms vanish for the chosen functions; this is standard for bound-state and scattering states with appropriate asymptotics). Doing so,

$$\int_0^\infty dr'\, \phi_f(r')\,(H-E)u_a(r') = \int_0^\infty dr'\, u_a(r')\,(H-E)\phi_f(r').$$

Consequently (15) can be rewritten as

$$(H-E)u_a(r) - \phi_f(r) \int_0^\infty dr'\, u_a(r')\,(H\phi_f)(r') = 0. \tag{16}$$

This coordinate-space formulation provides new physical insight into the projection mechanism and offers a direct implementation route for existing OPP codes, bridging the

gap between formal operator methods and practical numerical implementations. Equation (16) is the exact projected equation in coordinate space: the nonlocal kernel is $\phi_f(r)\,(H\phi_f)(r')$.

In most existing OPP applications ϕ_f is chosen as a Gaussian or oscillator $0s$ function. Then $(H\phi_f)(r')$ is replaced heuristically by $V(r')\phi_f(r')$, leading to the simplified kernel

$$(H-E)u_a(r) - \phi_f(r)\int_0^\infty dr'\,V(r')\phi_f(r')\,u_a(r') = 0. \tag{17}$$

Equation (17) is not the exact $\lambda\to\infty$ limit unless $(H_0\phi_f)(r') = 0$ or $H\phi_f = \varepsilon_f\phi_f$, which rarely holds for model ϕ_f.

Projection operators have been extensively employed in nuclear cluster models [1, 8]; however, the explicit coordinate-space derivation presented here provides a rigorous foundation for understanding when the commonly used approximate form (17) is valid, and how to implement the exact subtraction kernel when improved forbidden-state wavefunctions are available.

4 Effect on the resolvent and T-matrix: explicit Feshbach projection

The operator identity behind the elimination of the OPP parameter can be seen most clearly in the resolvent (Green's operator). Consider the OPP–modified Hamiltonian

$$H_\lambda = H + \lambda\,P_f, \qquad P_f = |\phi_f\rangle\langle\phi_f|, \qquad Q = 1 - P_f, \tag{18}$$

with P_f a rank-one projector. The resolvent of interest is

$$G_\lambda(E) = \frac{1}{E - H_\lambda}. \tag{19}$$

Following the Feshbach projection formalism [**?**], we derive the resolvent for the OPP-modified Hamiltonian. Our derivation makes explicit the λ-dependence and provides a clear physical interpretation of the $\lambda\to\infty$ limit.

Using the Feshbach projection operators P_f and Q, we decompose G_λ into block form

$$G_\lambda(E) = \begin{pmatrix} P_fG_\lambda P_f & P_fG_\lambda Q \\ QG_\lambda P_f & QG_\lambda Q \end{pmatrix}. \tag{20}$$

As in the standard Feshbach construction [5, 6], the effective Green's function in the Pauli-allowed space is the Schur complement

$$G_{aa}(E) = QG_\lambda(E)Q = \frac{1}{E - QHQ - QHP_f\,\frac{1}{E-P_fHP_f-\lambda}\,P_fHQ}. \tag{21}$$

Since $P_fHP_f = \langle\phi_f|H|\phi_f\rangle P_f \equiv \varepsilon_f P_f$, we obtain

$$G_{aa}(E) = \frac{1}{E - QHQ - QH|\phi_f\rangle\frac{1}{\varepsilon_f-E+\lambda}\langle\phi_f|HQ}. \tag{22}$$

The structure of Eq. (22) reveals why finite λ values in numerical calculations lead to residual forbidden-state coupling, scaling as $1/(\varepsilon_f - E + \lambda)$. This clarifies the convergence properties of practical OPP implementations.

The $\lambda \to \infty$ limit is now transparent:

$$\lim_{\lambda\to\infty} \frac{1}{\varepsilon_f - E + \lambda} = 0,$$

so the entire forbidden-space propagator collapses and we obtain the exact projected resolvent

$$G_{aa}^{(\infty)}(E) = \frac{1}{E - QHQ}. \tag{23}$$

Equation (23) is independent of λ and contains no forbidden admixtures: the dynamics is strictly confined to the Pauli-allowed subspace. When $\lambda \to \infty$, the forbidden space is effectively integrated out, and the reduced resolvent becomes λ-free, non-singular, correctly Pauli-projected. So the disappearance of explicit ϕ_f from $G_{aa}^{(\infty)}$ is a feature – it proves that the projection has been carried out fully, not approximately. In Eq. (23) all information on ϕ_f is contained implicitly in the projector $Q = 1 - |\phi_f\rangle\langle\phi_f|$. Thus, ϕ_f no longer appears as a dynamical degree of freedom; instead, it defines the reduced Hilbert space

$$\mathcal{H} \to Q\mathcal{H} = \{|u\rangle \in \mathcal{H} | \langle\phi_f|u\rangle = 0\}$$

in which the projected Hamiltonian operates.

The structure of (22) shows why taking λ very large suppresses forbidden components in numerical OPP calculations, but also explains the residual λ-dependence for finite values: the second term in (22) scales as $1/(\varepsilon_f - E + \lambda)$. Any finite λ leaves a small coupling to $|\phi_f\rangle$.

In momentum space, the separable-T-matrix expression (8) is obtained from the same Schur complement. Writing the full interaction in the extended basis $\{|\chi_i\rangle, |\phi_f\rangle\}$ produces a coupling matrix

$$\tilde{\boldsymbol{\lambda}} = \begin{pmatrix} \boldsymbol{\lambda} & 0 \\ 0^T & \lambda\,\mathbf{1} \end{pmatrix},$$

whose inverse tends to

$$\lim_{\lambda\to\infty} \tilde{\boldsymbol{\lambda}}^{-1} = \begin{pmatrix} \boldsymbol{\lambda}^{-1} & 0 \\ 0^T & 0 \end{pmatrix}.$$

The Schur complement applied to $[\tilde{\boldsymbol{\lambda}}^{-1} - \tilde{\boldsymbol{\mathcal{D}}}(E)]$ leads directly to Eq. (8). Thus the momentum-space elimination of λ is mathematically equivalent to the projected resolvent (23) in configuration space.

The projection operator $Q = 1 - P_f$ therefore captures the $\lambda \to \infty$ physics exactly: the OPP does not merely "push" forbidden states to high energy; it removes them entirely from the dynamical subspace. This clarifies why the λ-eliminated formulation reproduces observables obtained with very large λ, but without any residual numerical sensitivity to λ.

5 Numerical illustration: ^{6}Li binding energy

To demonstrate the practical implementation of explicit λ-elimination, we perform benchmark calculations of the ^{6}Li ground-state energy within the three-body $\alpha + p + n$ model. The α–nucleon interaction is represented by the Sack-Bidenharn-Breit (SBB) potential [22] in separable form, with a single S-wave forbidden state ϕ_f projected via the OPP term $\lambda_0|\phi_f\rangle\langle\phi_f|$. The nucleon–nucleon interaction employs the Malfliet-Tjon (MT)

potential [23] in its separable representation. These models offer an optimal balance between physical realism and computational efficiency for methodological testing.

We solved the bound-state Faddeev equations with two variants of the ^{4}He N interaction: (i) S- and P-waves only; (ii) S, P, and D partial waves. In each case the OPP strength λ_0 was increased from 10^0 to 10^5 MeV. The final line corresponds to the present $\lambda_0 \to \infty$ projected formulation, where the nonlocal subtraction kernel (8) is used and no large λ_0 appears numerically.

Table 1: Convergence of the ^{6}Li binding energy $E_{^6\mathrm{Li}}$ (MeV) as a function of the OPP strength λ_0 in the αN subsystem, with and without D waves.

λ_0 (MeV)	$E_{^6\mathrm{Li}}$ (S,P waves)	$E_{^6\mathrm{Li}}$ (S,P,D waves)
10^0	-4.146146	-4.249768
10^1	-4.148243	-4.251655
10^2	-4.148599	-4.252000
10^3	-4.148636	-4.252036
10^4	-4.148639	-4.252039
10^5	-4.148640	-4.252040
∞	-4.148640	-4.252040

The convergence pattern in Table 1 shows that finite–λ_0 OPP approaches the $\lambda_0 \to \infty$ limit only after several orders of magnitude increase in λ_0. For $\lambda_0 \lesssim 10^2$ MeV the binding energy differs from the limit by a few keV, as observed in earlier studies. With increasing λ_0 the value saturates, but continuation beyond $\lambda_0 \sim 10^4$ MeV provides no practical benefit and may introduce numerical stiffness.

In contrast, the projected formulation (last line in Table 1) produces the exact $\lambda_0 \to \infty$ result *without* tuning any large parameter. These results confirm that the analytic λ_0-elimination reproduces the asymptotic OPP limit and removes residual sensitivity to λ_0.

The example illustrates a common practical issue: although $\lambda_0 \gtrsim 10^4$ MeV is typically regarded as "large enough", the binding energy still displays a slow logarithmic convergence from $\lambda_0 = 10^1$ to 10^5 MeV. This behavior is consistent with the $1/(\varepsilon_f - E + \lambda_0)$ scaling in the Schur-complement expression (22). In multi-channel or three-body calculations the use of such large penalties can lead to ill-conditioning, "almost forbidden" states, and slow convergence in iterative solvers, as documented in Refs. [**?**, **?**, 11] etc. The λ_0-eliminated operator circumvents all such issues, while keeping exactly the same physical result.

The calculation in Table 1 is designed as a *methodological benchmark* for testing the λ–elimination scheme, not as a high-precision prediction. In the present implementation we (i) neglect the Coulomb interaction in the $\alpha-p$ pair and (ii) omit explicit three-body forces. Both approximations are known to affect the absolute binding of $A = 6$ systems; in particular, neglecting Coulomb induces extra attraction in the $\alpha-p$ channel and typically leads to overbinding in isospin-symmetric $\alpha + N + N$ calculations. The purpose here is to verify that the $\lambda \to \infty$ projected formulation reproduces the large-λ OPP limit *exactly* and removes residual λ-dependence in a controlled three-body setting. In future work we will incorporate the long-range α–p Coulomb interaction (e.g. via screening/renormalization or momentum-space contour methods) and realistic three-body forces; the projected operator carries over unchanged to those cases.

What the test demonstrates. (i) Finite-λ OPP shows the expected slow $1/(\varepsilon_f - E + \lambda)$ convergence over several decades of λ; (ii) the λ-eliminated projector reproduces the $\lambda \to \infty$ limit without tuning; (iii) the conclusion holds with and without D waves in the αN subsystem, indicating stability under partial-wave enrichment.

6 Discussion

As mentioned above, earlier works recognized that large λ suppresses forbidden components but did not write the closed λ-independent operator form. Finite λ was usually kept for three practical reasons:

1. *Approximate forbidden functions.* In many OPP and OCM implementations the forbidden functions ϕ_f are introduced explicitly, typically as harmonic-oscillator $0s$ relative-motion states between clusters. These ϕ_f are usually model functions rather than exact eigenstates of the microscopic Hamiltonian. Under such circumstances, keeping λ finite plays the role of a tunable "soft Pauli repulsion" compensating for this model dependence.

2. *Numerical stability.* Very large λ can cause ill-conditioned linear systems, especially in coupled channels or hyperspherical expansions. A moderate λ (e.g. 10^4–10^6 MeV) was a pragmatic choice.

3. *Computational convenience.* Writing and coding the exact nonlocal subtraction kernel was considered unnecessary when large-λ convergence appeared adequate.

Nevertheless, cases are known where the λ-dependence is non-negligible, e.g. in three-α calculations and the low-lying ^{6}He spectrum [11]. In such applications the exact formulation avoids parameter tuning and removes systematic uncertainties.

7 Conclusions

We have presented a comprehensive formulation for the analytic elimination of the OPP parameter λ that unifies momentum-space and configuration-space approaches. The key results are:

- The explicit Schur-complement T-matrix for general separable potentials (Eq. 8), providing a closed-form λ-free solution that can be directly implemented in momentum-space few-body calculations.
- A novel coordinate-space derivation yielding exact nonlocal subtraction kernels (Eqs. 16 and 17), clarifying the relationship between the exact projection and commonly used approximations.
- A clear physical interpretation of the λ-dependence through the resolvent analysis (Eq. 22), explaining the $1/(\varepsilon_f - E + \lambda)$ scaling observed in numerical implementations.
- Numerical demonstration in a three-body ^{6}Li benchmark showing that the λ-eliminated formulation reproduces the asymptotic large-λ limit without numerical tuning or convergence issues.

The configuration-space formulation is particularly valuable as it provides a direct route for implementing exact Pauli projection in existing OPP codes while offering new

physical insight into the projection mechanism. The unified treatment bridges formal operator methods with practical computational implementations.

The method is intended to complement, not replace, standard finite-λ practice. When ϕ_f are approximate model functions, finite λ may compensate residual imperfections. However, when ϕ_f accurately span the forbidden subspace, the present formulation provides a rigorous, parameter-free Pauli projection that eliminates convergence concerns and numerical stiffness associated with large λ values.

For future work, the separable momentum-space form is particularly suitable for Faddeev-equation treatments of α–n–n or α–p–n systems, while the configuration-space form enables direct implementation in coordinate-space OPP and OCM frameworks. The projection formalism naturally extends to include Coulomb interactions and three-body forces, providing a robust foundation for high-precision few-body calculations.

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