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Analysis of small-scale vertical perturbation modes against the background of a pulsating disk model

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This work investigates the problem of gravitational instability of small-scale vertical perturbation modes against the background of a pulsating composite model of disk-shaped self-gravitating systems. A non-stationary analog of the dispersion equation (NADE) has been derived in its general form for the vertical perturbation modes of the composite model. The results are presented in the form of critical diagrams of the initial virial ratio as a function of the model's rotation parameter for various values of the superposition parameter for each of the considered vertical perturbation modes. The analysis of the results shows, in particular, that the small-scale nature of the vertical perturbation modes plays a destabilizing role, while the system's rotation parameter, on the contrary, has a stabilizing effect.

Keywords: Gravitational instability; nonlinear non-stationary model; small-scale vertical perturbation modes

1 Introduction

It is well known that gravitational instability is a key mechanism in the formation of observable structures in self-gravitating systems. In the study of gravitational instability types, analytically solvable models of the objects being investigated play a special role. Since this work focuses on gravitational instabilities in disk-like self-gravitating systems (DSS), it is worth recalling that one of the analytically exact models of such collisionless systems is the Maclaurin equilibrium disk, first constructed by Bisnovatyi-Kogan and Zeldovich (Bisnovatyj-Kogan & Zel'dovich, 1970). The stability of this model has been studied by many authors (see, for example, (Kalnajs, 1972; Fridman et al., 1984; Abramyan, 1986; Bisnovatyi-Kogan, 1984, 1993; Binney & Tremaine, 2008; Fridman & Khoprskov, 2013)). However, real DSS are clearly non-stationary formations, and their global structure could have formed against the

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background of an early, nonlinear non-equilibrium stage of their evolution. Based on this, in Nuritdinov (1993, 2023), the Maclaurin disk was generalized to the nonlinear non-equilibrium case with radial non-stationarity, and the problem of the formation of primary structural formations was investigated in these models by studying gravitational instabilities of their corresponding large-scale perturbation modes (Nuritdinov et al., 2008, 2009; Mirtadjieva et al., 2011, 2022, 2025). However, the theory of stability for exact, analytically solvable models always requires refinement and generalization to more complex cases. Therefore, in (Mirtadjieva & Nuritdinov, 2012; Mirtadjieva, 2009, 2012a,b; Mirtadjieva & Mannapova, 2021), generalized and composite nonlinear non-stationary models of DSS with anisotropic velocity diagrams were constructed, and within these models, the issue of instabilities in observable structural modes of horizontal and vertical perturbations was studied. It should be noted that, until now, there has been almost no analysis of the instabilities of small-scale perturbation modes, not only in stationary but also in non-stationary DSS models. Based on this, in (Ganiev & Nuritdinov, 2022, 2021), we considered the behavior of small-scale tesseral modes $(m; N) = (12; 20), (14; 20), (16; 20), (18; 20)$ and sectoral modes $(m; N) = (10; 10), (11; 11), (12; 12), (13; 13), (14; 14), (15; 15)$ of horizontal perturbations against the background of an anisotropic non-stationary disk model. Here, N and m are the radial and azimuthal wave numbers, respectively. Unlike previous studies, the present work investigates the instabilities of small-scale vertical perturbation modes against the background of the composite model of a pulsating self-gravitating disk with an anisotropic velocity diagram developed by us.

2 A new composite model of a non-stationary disk and the NADE for vertical perturbation modes

In Mirtadjieva & Nuritdinov (2012), Nuritdinov was the first to construct a nonlinear non-stationary model for DSS that features an isotropic velocity diagram. This model of a self-gravitating disk was developed by generalizing the equilibrium model of Bisnovatyi-Kogan and Zeldovich (Mirtadjieva, 2009) to the case of radial pulsation.

$$\Psi_I\left(r, v_r, v_\perp, t\right) =$$

$$= \frac{\sigma_0}{2\pi\Pi\sqrt{1-\Omega^2}}\left[\frac{1-\Omega^2}{\Pi^2}\left(1-\frac{r^2}{\Pi^2}\right)-\left(v_r-v_a\right)^2-\left(v_\perp-v_c\right)^2\right]^{-1/2}\chi(\Pi-r). \quad (1)$$

Here, $R(t) = R_0\Pi(\mathrm{t})$ is the radius of the pulsating model (1) and, for convenience in calculations, we take $\mathrm{R}_0 = 1$. Further, Ω is a dimensionless parameter characterizing the degree of rotation of the disk $(0 \le \Omega \le 1)$. v_r and $v_\perp$ are the radial and tangential velocities of the particles, respectively $v_a = -\lambda r \sin\psi / \left(\sqrt{1-\lambda^2}\Pi^2\right), v_b = \Omega r/\Pi^2$, the function $\Pi(\psi) = \left(1-\lambda^2\right)^{-1}(1+\lambda\cos\psi)$ has the meaning of a stretch factor. Moreover, $\lambda = 1-(2T/|U|)_0$ is the pulsation amplitude, where $(2T/|U|)_0$ is the initial virial parameter, and χ is the Heaviside function. As is well known, a more realistic model should be considered as one that is anisotropic in terms of velocities and non-stationary. To achieve this, we use the method of averaging model (1) over the parameter Ω in the following form:

$$\Psi_A = \int_{-1}^{+1} \rho(\Omega) \cdot \Psi d\Omega, \quad \int_{-1}^{+1} \rho(\Omega) d\Omega = 1. \tag{2}$$

Here, $\rho(\Omega)$ is a weighting function of Ω. Taking the weighting function in the form:

$$\rho(\Omega) = \frac{8}{3\pi} \Omega^4 \left(1 - \Omega^2\right)^{-1/2}, \tag{3}$$

we obtained the following analytically solvable anisotropic model of a non-stationary disk:

$$\Psi_A = \frac{8\sigma_0}{3\pi} \left[\frac{1}{2\sqrt{(\mathrm{rv}_\perp + 1)^2 - k}} - \left(r^2 v_\perp^2 + 1\right) \right] \chi(k). \tag{4}$$

However, this model (4) is non-rotating, and to construct a rotating anisotropic pulsating model, we use the method of linear superposition of models (1) and (4) in the following form:

$$\Psi_{Comp}\left(r, v_r, v_\perp, t, v\right) = \nu \cdot \Psi_I + (1 - \nu) \cdot \Psi_A. \tag{5}$$

Substituting (1) and (4) into (5), we obtain the composite model in the following form:

$$\Psi_{Comp}\left(r, v_r, v_\perp, t, v\right) = \nu \cdot \frac{\sigma_0}{2\pi\Pi\sqrt{1 - \Omega^2}} \left[\frac{1 - \Omega^2}{\Pi^2} \left(1 - \frac{\mathrm{r}^2}{\Pi^2}\right) - \left(v_r - v_a\right)^2 - \right. \tag{6}$$

$$\left. - \left(v_\perp - v_b\right)^2 \right]^{-\frac{1}{2}} \cdot \chi(\mathrm{R} - \mathrm{r}) + (1 - \nu) \cdot \frac{8\sigma_0}{3\pi} \left[\frac{1}{2\sqrt{(\mathrm{rv}_\perp + 1)^2 - k}} - \left(r^2 v_\perp^2 + 1\right) \right] \chi(k).$$

Thus, the nonlinear pulsating rotating model (6) with an anisotropic velocity diagram that we constructed allows us to determine the most stable and unstable states over a wide range of possible initial conditions during the early stage of DSS.

It is worth noting that in (Mirtadjieva & Mannapova, 2021), against the background of the isotropic model (1), the NADE was derived in its general form for vertical perturbation modes as follows:

$$(1 + \lambda \cos\psi) \frac{d^2 D}{d\psi^2} + \left(\lambda \sin\psi + 2im\Omega\sqrt{1 - \lambda^2}\right) \frac{dD}{d\psi} + 2 \left[\gamma_N^m - 1 + \frac{m\Omega\sqrt{1 - \lambda^2}}{1 + \lambda\cos\psi} \times \right.$$

$$\left. \times \left(i\lambda \sin\psi - \frac{m\Omega\sqrt{1 - \lambda^2}}{2} \right) - \frac{\left(1 - \Omega^2\right)\left(1 - \lambda^2\right)\left(N^2 - m^2 + N - 2\right)}{6(1 + \lambda\cos\psi)} \right] D(\psi) = 0, \tag{7}$$

where

$$\gamma_N^m = \frac{(N + m)!!(N - m)!!}{(N + m - 1)!!(N - m - 1)!!}. \tag{8}$$

Multiplying (7) by the weighting function (3) and integrating the result over Ω from –1 to +1 , we obtain the NADE within the framework of the anisotropic model (4) in the following form:

$$(1 + \lambda \cos\psi) \frac{d^2 D}{d\psi^2} + \lambda \sin\psi \frac{dD}{d\psi} +$$

$$+2\left[\gamma_N^m-1-\frac{\left(1-\lambda^2\right)\left(N^2+N+14m^2-2\right)}{36(1+\lambda\cos\psi)}\right]D(\psi)=0\,. \tag{9}$$

Now, based on the method of linear superposition of equations (7) and (9) for the two non-stationary models, isotropic (1) and anisotropic (4), we obtain the NADE for the composite model (6) as follows:

$$\text{NADE}_{\text{Comp}}=\nu\cdot\text{NADE}_I+(1-\nu)\cdot\text{NADE}_A\,. \tag{10}$$

Substituting (7) and (9) into (10), we obtain the NADE for the composite model (8) in its general form as follows:

$$(1+\lambda\cos\psi)\frac{d^2D}{d\psi^2}+\left(\lambda\sin\psi+2\nu im\Omega\sqrt{1-\lambda^2}\right)\frac{dD}{d\psi}+2\left[\gamma_N^m-1+\nu\frac{m\Omega\sqrt{1-\lambda^2}}{1+\lambda\cos\psi}\times\right.$$
$$\times\left(i\lambda\sin\psi-\frac{m\Omega\sqrt{1-\lambda^2}}{2}\right)-\nu\frac{\left(1-\Omega^2\right)\left(1-\lambda^2\right)\left(N^2-m^2+N-2\right)}{6(1+\lambda\cos\psi)}- \tag{11}$$
$$\left.-(1-\nu)\frac{\left(1-\lambda^2\right)\left(N^2+N+14m^2-14\right)}{36(1+\lambda\cos\psi)}\right]D(\psi)=0.$$

Thus, using the NADE (11), it is possible to study any type of vertical perturbation modes against the background of the composite non-stationary disk model.

3 Results of calculations

In this work, we investigate the problem of gravitational instability for small-scale vertical perturbation modes m = 1; N = 10 and m = 1; N = 12 using the NADE (11) of the composite model at values of the superposition parameter $\nu = 0.00, 0.50,$

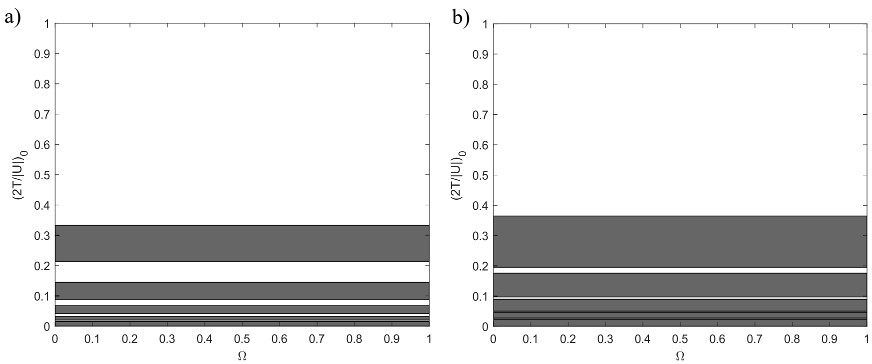

Figure 1 Critical dependence of the initial virial ratio on the rotation parameter of the composite model (6) for vertical perturbation modes a) $m = 1; N = 10$, b) $m = 1; N = 12$ at $\nu = 0.0$. The shaded area corresponds to the instability zone.

and 1.00 . The results of the numerical calculations of the NADE for the considered perturbation modes are presented in the form of critical diagrams of the initial virial ratio as a function of the disk's rotation parameter, and the corresponding instability increments are also calculated. In Fig. 1, the critical diagrams of the initial virial ratio as a function of the rotation parameter of the non-stationary disk (6) are shown for both perturbation modes at the superposition parameter $v = 0.0$. As can be seen from these diagrams, as the degree of small-scale nature of the vertical perturbation modes increases, the instability regions also expand.

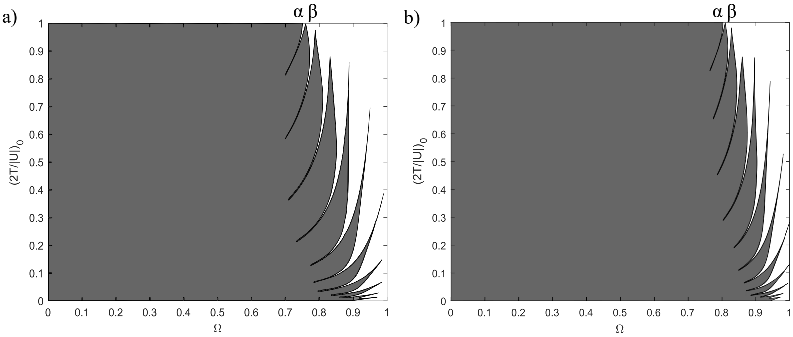

Figure 2 Critical dependence of the initial virial ratio on the rotation parameter of the composite model (6) for vertical perturbation modes: a) $m = 1; N = 10, m = 1; N = 12$, at $\nu = 0.5$. Here: $\alpha = 0.751$, $\beta = 0.760$ b) $\alpha = 0.803$, $\beta = 0.809$.

In the critical diagrams shown for the superposition parameter $\nu = 0.5$, it can also be observed that as the value of N increases, the instability region expands (Fig. 2). Furthermore, it is worth noting that in the interval of disk rotation parameter values $0.0 \leq \Omega < 0.7$, the composite model is completely unstable with respect to these two vertical perturbation modes.

Figure 3 shows the critical diagrams for the case of the superposition parameter $\nu = 1.0$. Here, the degree of small-scale nature of the vertical perturbation modes

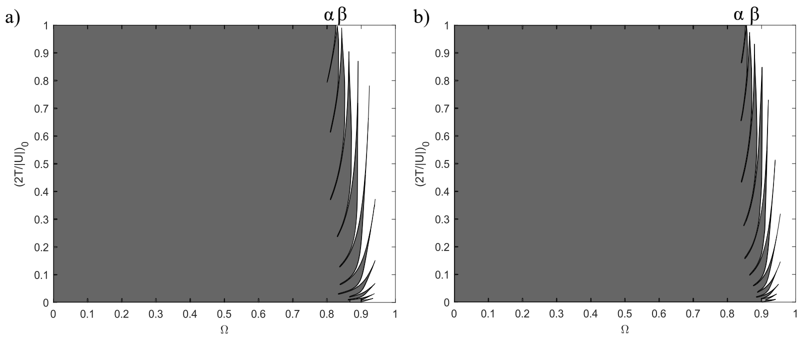

Figure 3 Critical dependence of the initial virial ratio on the rotation parameter of the composite model (6) for vertical perturbation modes: a) $m = 1; N = 10$, b) $m = 1; N = 12$, at $\nu = 1.0$. Here: a) $\alpha = 0.827$, $\beta = 0.830$ b) $\alpha = 0.853$, $\beta = 0.856$

plays a destabilizing role, while the system's rotation parameter, on the contrary, has a stabilizing effect. Thus, it can be concluded that this behavior of the critical diagrams obtained for the vertical perturbation modes is completely opposite to the situations observed in the critical diagrams for horizontal modes (Ganiev & Nuritdinov, 2022, 2021).

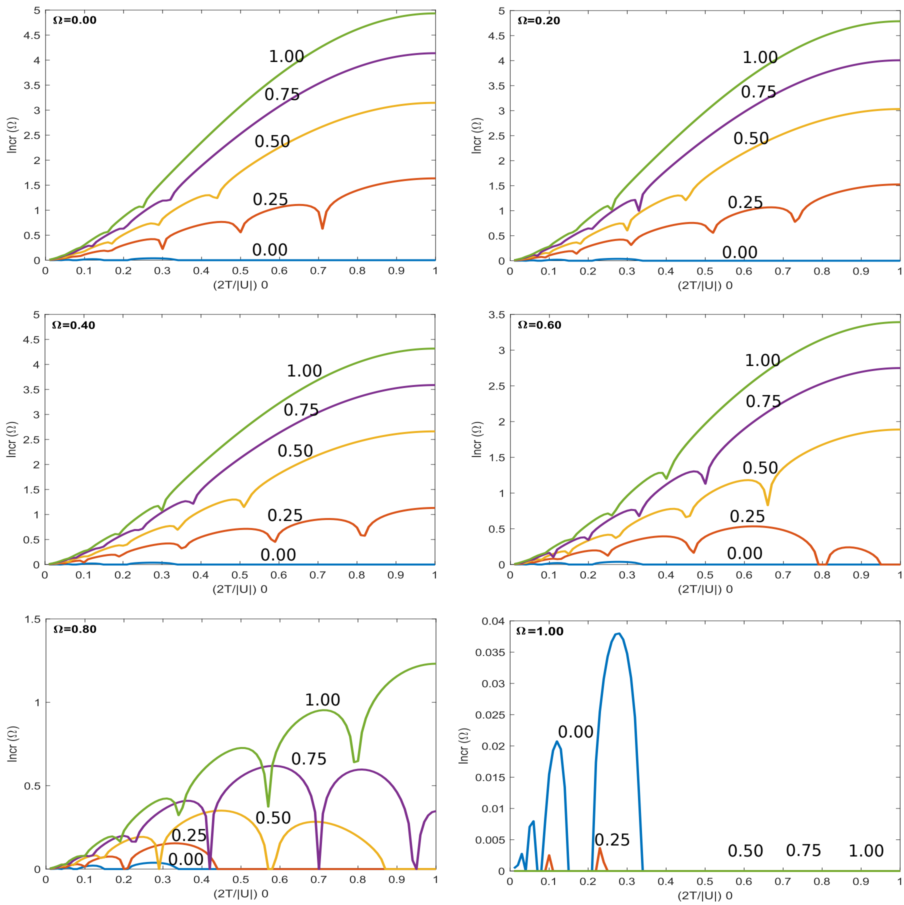

Figure 4 Dependencies of instability increments on the initial virial ratio for the vertical perturbation mode $m = 1; N = 10$ at different values of the rotation and superposition parameters of the composite model.

In the next figure (Fig. 4), the results of comparing the instability increments of vertical perturbation modes $\mathrm{m} = 1; \mathrm{N} = 10$ at different values of the rotation and superposition parameters of the composite model (8) are presented. Here, turning to the superposition parameter ν, it is noted that as the disk transitions from an isotropic to a purely anisotropic state, the instability increment of the composite model smoothly decreases at small and moderate values of the rotation parameter Ω. However, when the rotation parameter approaches its maximum value, the opposite behavior is observed.

From Fig. 5, it is evident that the pattern repeats similarly to the case of the (1; 10) mode. Additionally, it should be noted that as the value of the rotation parameter increases, both the rate of instability and the range of critical values of the initial virial ratio of the composite model gradually decrease.

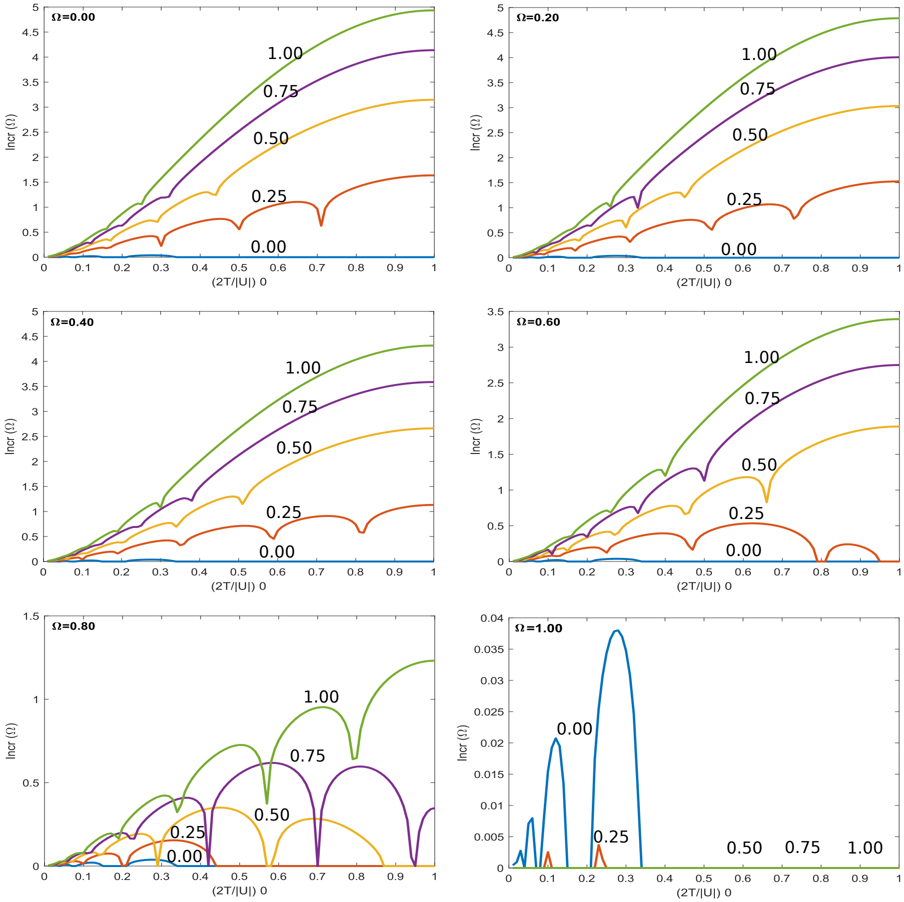

Figure 5 Dependencies of instability increments on the initial virial ratio for the vertical perturbation mode $m = 1; N = 12$ at different values of the rotation and superposition parameters of the composite model.

4 Conclusion

A pulsating composite model of a non-stationary self-gravitating disk has been constructed using the method of linear superposition of two nonlinear non-stationary models: one isotropic and one anisotropic. A non-stationary analog of the dispersion equation (NADE) has been derived in a general form for vertical perturbation modes of the composite model, and based on it, numerical calculations were carried out for small-scale vertical perturbation modes with m = 1; N = 10 and m = 1; N = 12. Critical diagrams of the initial virial ratio $(2\ T/\mid U)_0$ versus the disk rotation parameter Ω have been constructed for various values of the model's superposition parameter ν, and the instability increments for the given vertical perturbation modes have also been computed. The results show that as the radial wave number of the vertical perturbation modes increases, the instability region on the critical diagrams also expands. Thus, the degree of small-scale character of the vertical modes plays a destabilizing role, while the rotation parameter of the system provides a stabilizing effect. It has been shown that in the range $0.0 \leq \Omega < 0.7$, the composite model is completely unstable with respect to these two vertical perturbation modes. Referring to the superposition parameter ν, it is observed that during the transition from the isotropic to the purely anisotropic state of the disk, the instability increment of the composite model gradually decreases at low and moderate values of the rotation parameter Ω. However, as the rotation parameter approaches its maximum value, the opposite trend is observed. It is demonstrated that with an increase in the rotation parameter, both the instability rate and the range of critical values of the initial virial ratio of the composite model gradually decrease.

References

Abramyan, M. G. 1986, Dynamics of Embedded Gravitating Systems, Doctoral Dissertation Abstract, Erevan

Bisnovatyj-Kogan, G. S., & Zel'dovich, Ya. B. 1970, Astrofizika, 6, 387

Bisnovatyi-Kogan, G. S. 1984, Astrophysics, 20(3), 302

Bisnovatyi-Kogan, G. S. 1993, Astrophysics, 35(2–3), 410

Binney, J., & Tremaine, S. 2008, Galactic Dynamics, 2nd ed.

Fridman, A. M., Polyachenko, V. L., Aries, A. B., & Poliakoff, I. N. 1984, Physics of gravitating systems. II. Nonlinear collective processes. Astrophysical applications.

Fridman, A. M., & Khoprskov, A. V. 2013, Physics of Galactic Disks, Cambridge International Science Publishing

Ganiev, J., & Nuritdinov, S. 2021, Uzbek Journal of Physics, 23(2), 1

Ganiev, J., & Nuritdinov, S. 2022, Open Astronomy, 31(1), 92

Kalnajs, A. J. 1972, ApJ, 175, 63

Mirtadjieva, K. T. 2009, Gravitation and Cosmology, 15(3), 278

Mirtadjieva, K. T. 2012a, Gravitation and Cosmology, 18(1), 6

Mirtadjieva, K. T. 2012b, Gravitation and Cosmology, 18(4), 249

Mirtadjieva, K. T., & Nuritdinov, S. N. 2012, Astrophysics, 55(4), 551

Mirtadjieva, K. T., Nuritdinov, S. N., Ruzibaev, J. K., & Khalid, M. 2011, Astrophysics, 54(2), 184

Mirtadjieva, K. T., Nuritdinov, S. N., Mannapova, K. A., & Sadibekova, T. O. 2022, Astrophysics, 65(2), 247

Mirtadjieva, K. T., Nuritdinov, S. N., Botirov, F. U., & Ganiev, J. M. 2025, Gravitation and Cosmology, 31(1), 46

Mirtadjieva, K. T., & Mannapova, K. A. 2021, Gravitation and Cosmology, 27(3), 212

Nuritdinov, S. N. 1993, Author's abstract of dissertation for doctor of phys.-math. sciences

Nuritdinov, S. N. 2023, Nonlinear Waves and Oscillations in Self-Gravitating Disk-like Systems (in Russian), Tashkent, 163 pp.

Nuritdinov, S. N., Mirtadjieva, K. T., & Sultana, M. 2008, Astrophysics, 51(3), 410

Nuritdinov, S. N., Mirtadjieva, K. T., Ahmad, I., & Ruzibaev, J. K. 2009, Astrophysics, 52(4), 584