On the Instabilities of Small-Scale Modes of Oscillations Against the Background of a Collapsing Galaxy Model

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Abstract—In this work, the gravitational instability of small-scale perturbations with an azimuthal wave number $m = 2$ in disk-like self-gravitating systems is considered. Calculations of horizontal small-scale oscillation modes $(m; N) = (2;10)$ and $(2; 20)$ against the background of a nonlinearly non-equilibrium anisotropic model of a self-gravitating disk are performed. Critical diagrams of the relationship between the virial parameter and the degree of rotation for these modes are constructed, and the increments of instability for different values of the rotation parameter are calculated. The results show that the instability for the oscillation mode $(2; 10)$ begins at a virial parameter value of $(2T/|U|)_0 \approx 0.217$ at $\Omega = 0$ and reaches 0.413 at $\Omega = 1$. For the oscillation mode $(2; 20)$, the instability starts at a virial parameter value of $(2T/|U|)_0 \approx 0.128$ at $\Omega = 0$ and reaches 0.146 at $\Omega = 1$. It is found that with an increase in the rotation parameter, the instability region also increases, while with an increase in the degree of small-scale structure, the instability region significantly decreases. The work is partially based on a talk presented at the Modern Stellar Astronomy 2024 conference.

1. INTRODUCTION

It is well known that the theory of the spiral structure of galaxies considers an equilibrium disk-like model as the initial state [1–4]. However, in reality, any perturbation must develop against the background of a nonstationary model [5–7]. At present, several nonlinear nonstationary isotropic and anisotropic models exist for studying large-scale formations in the disk subsystems of self-gravitating objects [8–13]. Corresponding nonstationary dispersion equations (NDEs) have been derived for each perturbation mode. However, the role of small-scale modes in disks, particularly in the context of nonstationary disks, has not yet been studied. This is why the present work investigates small-scale oscillations in the context of a nonlinear nonstationary pulsating model, where the azimuthal wavenumber is $m = 2$, and the radial wavenumber is $N = 10$ and 20.

2. MODEL

The study of the structure and evolution of disk-like self-gravitating systems requires an analysis of the instability not only of large-scale oscillation modes but also of various small-scale perturbations. Investigating the mechanisms behind the formation of small-scale structural formations requires constructing a mathematical model of a nonstationary self-gravitating disk with an anisotropic velocity distribution. Based on our model, a well-known isotropic nonlinear pulsating model was constructed [5]:

$$\Psi(r, v_r, v_\perp, t) = \frac{\sigma_0}{2\pi\Pi\sqrt{1-\Omega^2}}$$

$$\times \left[\frac{1-\Omega^2}{\Pi^2}\left(1 - \frac{r^2}{\Pi^2}\right) - \left(v_r - v_a\right)^2 \right. \tag{1}$$

$$\left. - \left(v_\perp - v_b\right)^2 \right]^{-1/2} \chi(R - r).$$

Here, $\sigma_0 = \sigma(0;0)$, Ω is the dimensionless parameter characterizing the degree of rotation of the rigid-body disk $(0 \le \Omega \le 1)$; v_r and $v_\perp$ are the radial and tangential particle velocities, χ is the Heaviside function, function $\Pi(t)$ has a sense of as a stretching coefficient,

$\Pi(t) = (1 + \lambda \cos \psi)(1 - \lambda^2)^{-1}$, $t = (\psi + \lambda \sin \psi) \times$ $(1 - \lambda^2)^{-3/2}$, $v_a = -\lambda \sqrt{1 - \lambda^2}(r \sin \psi / \Pi^2)$, $v_b = \Omega r / \Pi^2$. The model pulsates with an amplitude $\lambda = 1 - (2T/|U|)_0$, where $(2T/|U|)_0$ is the virial parameter. Let us construct a pulsating model with an anisotropic velocity distribution by averaging over the parameter Ω:

$$\Psi_{\text{anis}} = \frac{\int\limits_{-1}^{+1} \Psi \rho(\Omega) d\Omega}{\int\limits_{-1}^{+1} \rho(\Omega) d\Omega}. \tag{2}$$

Here, $\rho(\Omega)$ is a weighting function of Ω. We constructed the anisotropic model by taking the weighting function in the form $\rho(\Omega) = (2/\pi)(1 - \Omega^2)^{1/2}(1 + \Omega\Omega')$. After integrating over Ω and performing some transformations, we obtain a nonlinear pulsating model with an anisotropic velocity distribution:

$$\Psi_{\text{anis}} = \frac{\sigma_0}{\pi}[1 + \Omega'(xv_y - yv_x)]$$
$$\times \chi\left(\left(1 - \frac{r^2}{\Pi^2}\right)(1 - \Pi^2 v_\perp^2) - \Pi^2(v_r - v_a)^2\right). \tag{3}$$

It can be easily shown that Ω' plays the role of the angular velocity of the rotating anisotropic disk. For model (3), we derived the general-form nonstationary dispersion equation (NDE) using Nuritdinov's general principle [5]:

$$a_{\text{anis}}(\psi) = \frac{4}{N(N^2 - 1)\Pi^3} \gamma_m^N (N - 1)$$
$$\times (N^2 + N - m^2)\bar{D}_N + i\Omega' m(N^2 + N - m^2)d_N, \tag{4}$$

where

$$\gamma_N^m = \frac{(N + m - 1)!!(N - m - 1)!!}{(N + m)!!(N - m)!!},$$

$$\bar{D}_N = \int\limits_{-\infty}^{\psi} EW^{N-1} \frac{dP_N}{d(\cosh)} d\psi_1,$$

$$d_N = \int\limits_{-\infty}^{\psi} EI_N d\psi_1,$$

$$E = \Pi^3(\psi_1)S(\psi, \psi_1)a(\psi_1),$$

$$I_N = W^{N-1}\left[\frac{1}{N + 2}\frac{d^2 P_{N+1}}{d(\cosh)^2} - \frac{P_N}{d(\cosh)}\right]\tanh,$$

$$W = \frac{1 + \lambda \cos \psi_1}{1 + \lambda \cos \psi},$$

$$\cosh = \frac{(\cos \psi + \lambda)(\cos \psi_1 + \lambda) + (1 - \lambda^2)\sin \psi \sin \psi_1}{(1 + \lambda \cos \psi)(1 + \lambda \cos \psi_1)}.$$

$P_N(\cosh)$ is the Legendre polynomial, and N is the primary perturbation index. Based on equation (4), we analyzed several specific small-scale sectoral [14] and tesseral [15] perturbation modes in the pulsating disk model (3). We constructed critical diagrams of the virial parameter versus rotation degree for each of these modes and calculated the corresponding instability increments.

3. CALCULATION RESULTS

We obtained the corresponding NDE for the case of $m = 2$ and an arbitrary radial wavenumber N. However, it is difficult to determine the asymptotic behavior of this NDE when N approaches infinity. Therefore, we incrementally increase the N value and consider the physics of specific cases. In this work, we present the calculation results for the disk model in small-scale perturbation modes, particularly for the wavenumbers $m = 2$; $N = 10$ and 20.

Let us consider the small-scale mode $m = 2$, $N = 10$. Substituting these values into equation (4), we obtain the following NDE for this oscillation mode:

$$a_{2;10}(\psi) = \frac{49}{32\,768\Pi^3}[159\bar{D}_{10} + i\Omega35d_{10}]. \tag{5}$$

Proceeding to the differential form, we have the following:

$$(1 + \lambda \cos \psi)\frac{d^2 l_\tau(\psi)}{d\psi^2} + \lambda \sin \psi \frac{dl_\tau(\psi)}{d\psi} + l_\tau(\psi)$$
$$= \frac{(\lambda + \cos \psi)^{9-\tau} \sin^\tau \psi}{(1 + \lambda \cos \psi)^{18}} K_{2;10}(\psi). \tag{6}$$

Here,

$$l_\tau(\psi) = \int\limits_{-\infty}^{\psi} (1 + \lambda \cos \psi_1)^3 S(\psi, \psi_1) a_{2;10}(\psi_1)$$
$$\times (\lambda + \cos \psi_1)^{9-\tau} \sin^\tau \psi_1 d\psi_1,$$

the subscript takes values from $\tau = 0$ to $\tau = 9$, where λ is the pulsation amplitude of the disk. The expressions for $K_{2;10}(\psi)$ are given in Appendix I.

Numerical calculations of the NDE show that for the oscillation mode $m = 2$; $N = 10$ instability begins at a virial parameter value of $(2T/|U|)_0 \approx 0.217$ and reaches 0.413. There is a very narrow stability region between the values $(2T/|U|)_0 \approx 0.2174{-}0.2178$, while a clearly visible "peninsula" of stability appears between $(2T/|U|)_0 \approx 0.2379{-}0.2312$ and $(2T/|U|)_0 \approx 0.375{-}0.397$ (Fig. 1). From this, we observe that as
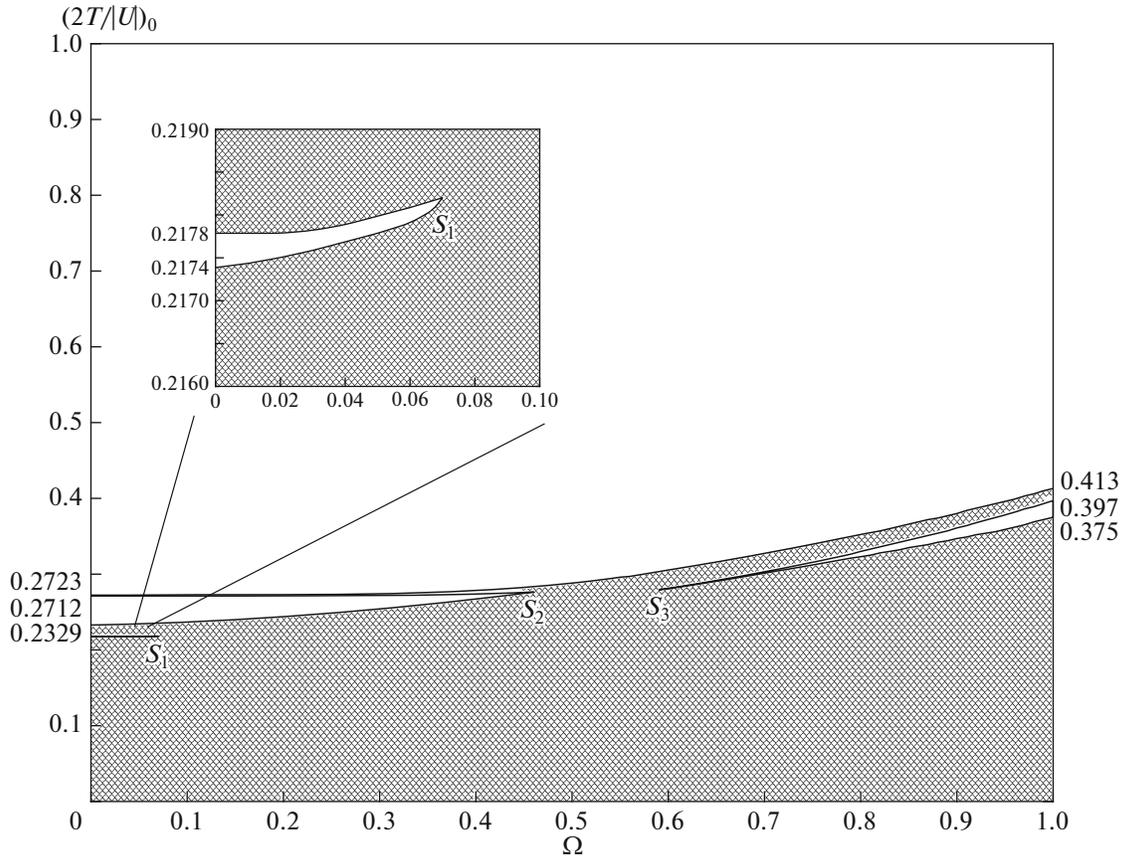
Fig. 1. Critical dependence of the initial virial ratio on the rotation parameter of model (3) for the oscillation mode $m = 2$, $N = 10$. Here, $S_1 = (0.07; 0.2181)$, $S_2 = (0.46; 0.2763)$ and $S_3 = (0.59; 0.2793)$. The instability region is shaded.
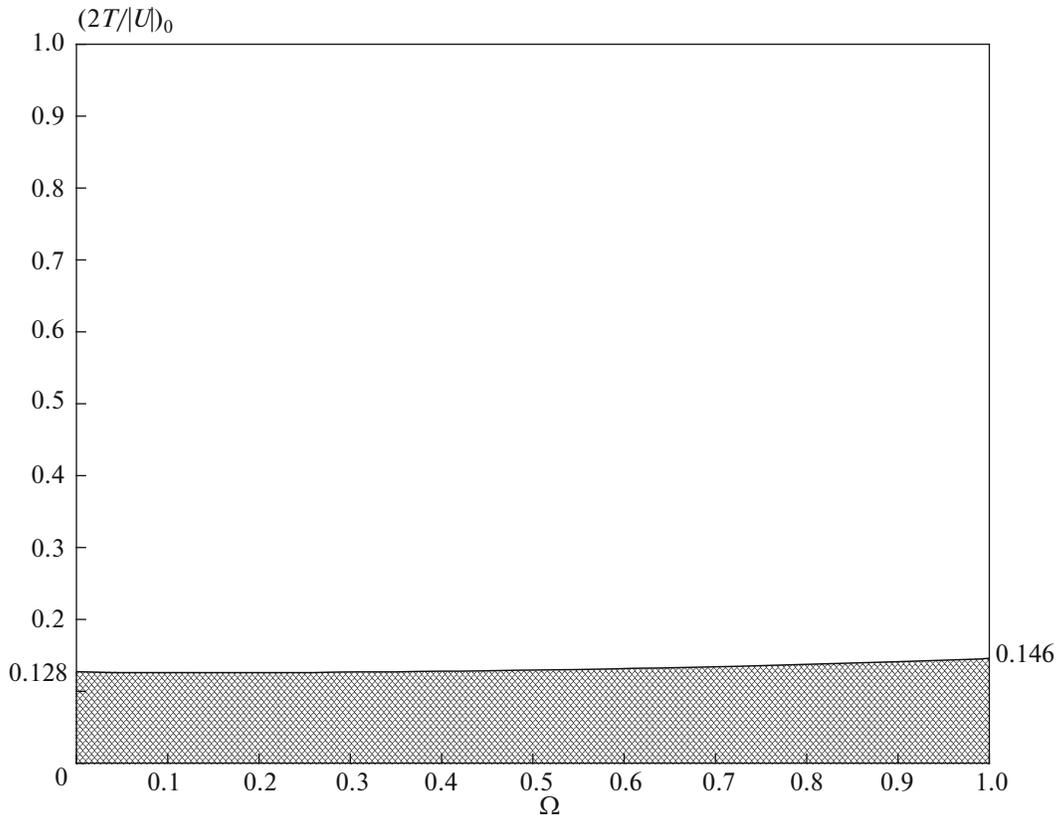
Fig. 2. Critical dependence of the virial ratio on the rotation parameter for $m = 2$; $N = 20$.
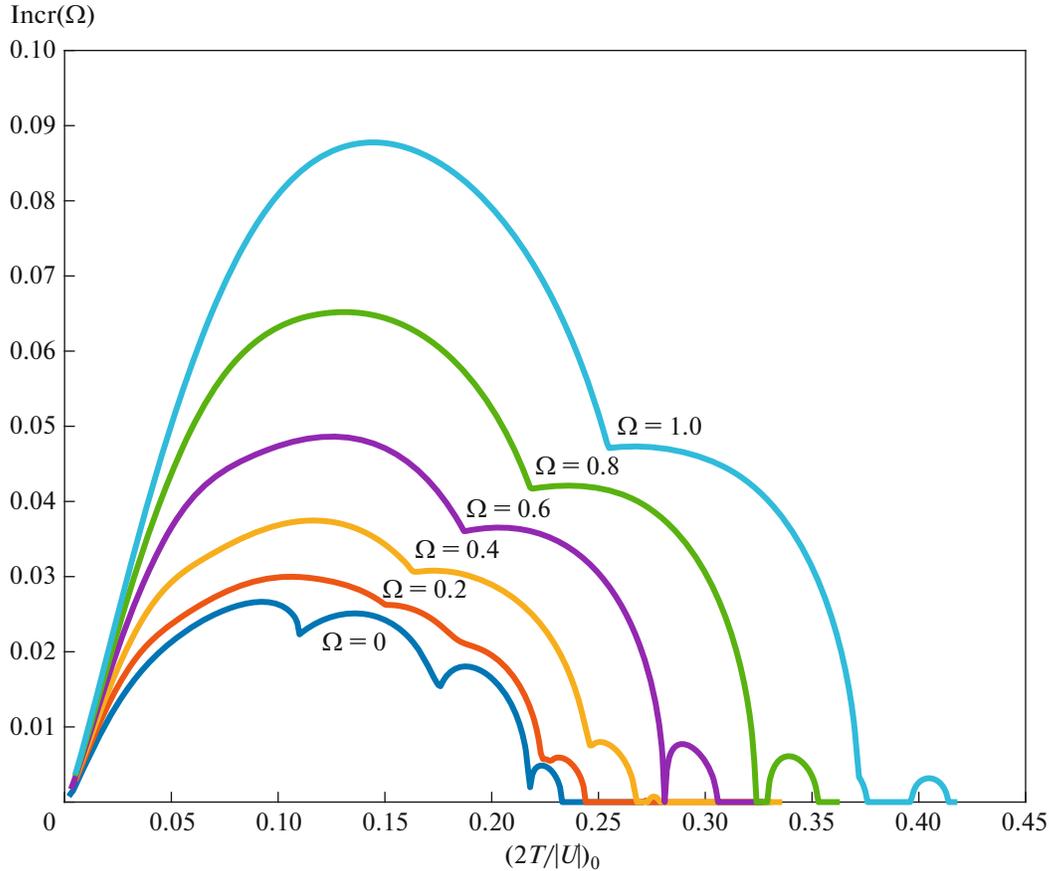
Fig. 3. Comparison of instability increments for $m = 2$; $N = 10$ at different values of the rotation parameter.

the rotation parameter increases, the instability region grows significantly.

The NDE for the oscillation mode $m = 2$; $N = 20$ has the following form:

$$a_{2;20}(\psi) = \frac{537\,251}{17\,179\,869\,184\Pi^3}[3952\bar{D}_{20} + i\Omega 415 d_{20}]. \quad (7)$$

Proceeding to the differential form, we have the following:

$$(1 + \lambda\cos\psi)\frac{d^2 l_\tau(\psi)}{d\psi^2} + \lambda\sin\psi\frac{d l_\tau(\psi)}{d\psi} + l_\tau(\psi)$$

$$= \frac{(\lambda + \cos\psi)^{19-\tau}\sin^\tau\psi}{(1 + \lambda\cos\psi)^{38}}M_{2;20}(\psi). \quad (8)$$

Now

$$l_\tau(\psi) = \int_{-\infty}^{\psi}(1 + \lambda\cos\psi_1)^3 S(\psi,\psi_1) a_{2;20}(\psi_1)$$

$$\times (\lambda + \cos\psi_1)^{19-\tau}\sin^\tau\psi_1 d\psi_1,$$

and subscript τ takes the values from $\tau = 0$ to $\tau = 19$. The expressions for $M_{2;20}(\psi)$ are rather cumbersome, so we have included them in Appendix II.

Instability for the oscillation mode $m = 2$; $N = 20$ begins at a virial parameter value of $(2T/|U|)_0 \approx 0.128$ and extends to 0.146 (Fig. 2). In this case, no stability islands are observed on the diagram. The shaded area is clearly smaller than the instability region in Fig. 1. Therefore, as N increases, the instability region in terms of the virial parameter narrows.

A comparison of instability increments (Fig. 3) shows that the curves for the oscillation modes $m = 2$; $N = 10$ are arranged in descending order of the rotation parameter and do not intersect. In the case of $m = 2$; $N = 20$ the curves also follow a decreasing order of the rotation parameter (Fig. 4). For $\Omega = 0.0-0.2$ and $\Omega = 0.4-1.0$, intersections of the curves are observed separately.

4. CONCLUSIONS

In this study, we examined the instabilities of oscillation modes $m = 2$, $N = 10$ and 20. The calculation
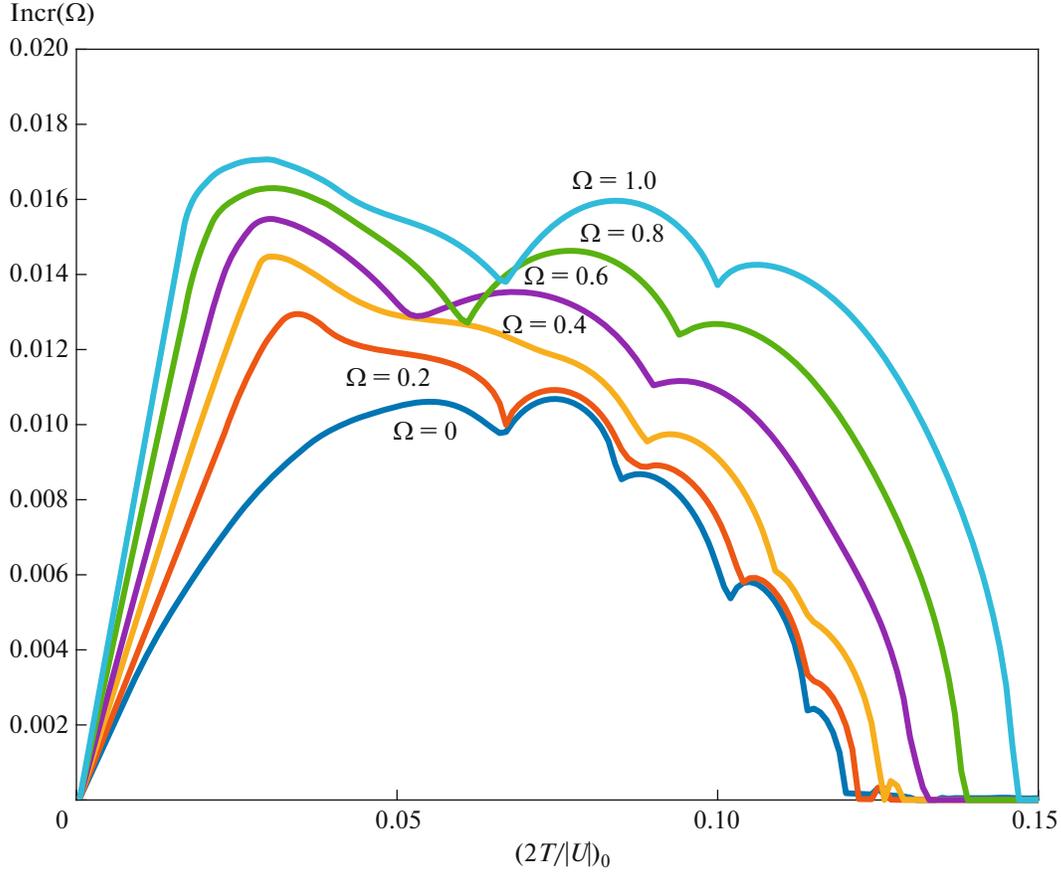
Fig. 4. Comparison of instability increments for $m = 2$; $N = 20$ at different values of the rotation parameter.

results indicate that the instability of the perturbation mode (2; 10) begins at a virial parameter value of $(2T/|U|)_0 \approx 0.233$ and extends to 0.413. There is a very narrow stability region between the values $(2T/|U|)_0 \approx 0.2174-0.2178$, and a distinctly visible stability island appears between $(2T/|U|)_0 \approx 0.2379-0.2312$ and $0.375-0.397$.

The stability region for the oscillation mode (2; 20) is wider than in the case of $N = 10$ and begins at a virial parameter value of $(2T/|U|)_0 \approx 0.128$.

It was found that the instability region grows with an increasing rotation parameter and decreases with an increase in the small-scale structure.

The comparison of Figs. 3 and 4 shows that the instability increments for the oscillation modes $m = 2$; $N = 20$ are significantly smaller than in the case of $m = 2$; $N = 10$. In the case of $N = 20$ same as with $N = 10$, the curves are arranged in descending order of the rotation parameter. Between the values $\Omega = 0.0-0.2$ and $\Omega = 0.4-1.0$, intersections of the curves are observed separately.

APPENDIX I

$$K_{2;10}(\psi) = \frac{49}{196\,608}\left\{954\left[\left(55c^9 - 495c^7e^2b^2\right.\right.\right.$$

$$\left.+ \frac{3465}{4}c^5e^4b^4 - \frac{5775}{16}c^3e^6b^6 + \frac{3465}{128}ce^8b^8\right)l_0(\psi)$$

$$+ \left(1485e^2bc^8 - 6930e^4b^3c^6 + \frac{51975}{8}e^6b^5c^4\right.$$

$$\left.- \frac{10395}{8}e^8b^7c^2 + \frac{3465}{128}e^{10}b^9\right)l_1(\psi)$$

$$+ \left(\frac{28215}{2}e^4b^2c^7 - \frac{530145}{16}e^6b^4c^5 - 495c^9e^2\right.$$

$$\left.+ \frac{509355}{32}e^8b^6c^3 - \frac{10395}{8}e^{10}b^8ct\right)l_2(\psi)$$

$$+ \left(\frac{234465}{4}e^6b^3c^6 - \frac{557865}{8}e^8b^5c^4 - 6930e^4bc^8\right.$$

$$\left.\left.+ \frac{509355}{32}e^{10}b^7c^2 - \frac{5775}{16}e^{12}b^9\right)l_3(\psi)\right.$$

$$+ \left(\frac{7\,494\,795}{64} e^8 b^4 c^5 - \frac{557\,865}{8} e^{10} b^6 c^3 - \frac{530\,145}{16} e^6 b^2 c^7 \right.$$

$$\left. + \frac{51\,975}{8} e^{12} b^8 c + \frac{3465}{4} c^9 e^4 \right) l_4(\psi)$$

$$+ \left(\frac{7\,494\,795}{64} e^{10} b^5 c^4 - \frac{530\,145}{16} e^{12} b^7 c^2 - \frac{557\,865}{8} e^8 b^3 c^6 \right.$$

$$\left. + \frac{3465}{4} e^{14} b^9 + \frac{51\,975}{8} e^6 b c^8 \right) l_5(\psi)$$

$$+ \left(\frac{234\,465}{4} e^{12} b^6 c^3 - 6930 e^{14} b^8 c - \frac{557\,865}{8} e^{10} b^4 c^5 \right.$$

$$\left. + \frac{509\,355}{32} e^8 b^2 c^7 - \frac{5775}{16} c^9 e^6 \right) l_6(\psi)$$

$$+ \left(\frac{28\,215}{2} e^{14} b^7 c^2 - 495 e^{16} b^9 - \frac{10\,395}{8} e^8 b^8 \right.$$

$$\left. - \frac{530\,145}{16} e^{12} b^5 c^4 + \frac{509\,355}{32} e^{10} b^3 c^6 \right) l_7(\psi)$$

$$+ \left(1485 e^{16} b^8 c - 6930 e^{14} b^6 c^3 + \frac{51\,975}{8} e^{12} b^4 c^5 \right.$$

$$\left. - \frac{10\,395}{8} e^{10} b^2 c^7 + \frac{3465}{128} c^9 e^8 \right) l_8(\psi)$$

$$+ \left(55 e^{18} b^9 - 495 e^{16} b^7 c^2 + \frac{3465}{4} e^{14} b^5 c^4 \right.$$

$$\left. - \frac{5775}{16} e^{12} b^3 c^6 + \frac{3465}{128} e^{10} b c^8 \right) l_9(\psi) \Bigg]$$

$$+ 840 i e \Omega \Bigg[\left(- \frac{495}{4} b c^8 + \frac{1155}{2} e^2 b^3 c^6 - \frac{17\,325}{32} e^4 b^5 c^4 \right.$$

$$\left. + \frac{3465}{32} e^6 b^7 c^2 - \frac{1155}{512} e^8 b^9 \right) l_0(\psi)$$

$$+ \left(\frac{495}{4} c^9 - \frac{5445}{2} c^7 e^2 b^2 + \frac{197\,505}{32} c^5 e^4 b^4 \right.$$

$$\left. - \frac{93\,555}{32} c^3 e^6 b^6 + \frac{121\,275}{512} c^8 b^8 \right) l_1(\psi)$$

$$+ \left(- \frac{308\,385}{16} e^4 b^3 c^6 + \frac{5445}{2} e^2 b c^8 + \frac{696\,465}{32} e^6 b^5 c^4 \right.$$

$$\left. - \frac{620\,235}{128} e^8 b^7 c^2 + \frac{3465}{32} e^{10} b^9 \right) l_2(\psi)$$

$$+ \left(- \frac{1867\,635}{32} e^6 b^4 c^5 + \frac{308\,385}{16} e^4 b^2 c^7 \right.$$

$$+ \frac{4163\,775}{128} e^8 b^6 c^3 - \frac{1155}{2} c^9 e^2 - \frac{93\,555}{32} e^{10} b^8 c \Bigg) l_3(\psi)$$

$$+ \left(- \frac{21\,417\,165}{256} e^8 b^5 c^4 + \frac{1867\,635}{32} e^6 b^3 c^6 \right.$$

$$+ \frac{696\,465}{32} e^{10} b^7 c^2 - \frac{197\,505}{32} e^4 b c^8 - \frac{17\,325}{32} e^{12} b^9 \Bigg) l_4(\psi)$$

$$+ \left(- \frac{1867\,635}{32} e^{10} b^6 c^3 + \frac{21\,417\,165}{256} e^8 b^4 c^5 \right.$$

$$+ \frac{197\,505}{32} e^{12} b^8 c - \frac{696\,465}{32} e^6 b^2 c^7 + \frac{17\,325}{32} c^9 e^4 \Bigg) l_5(\psi)$$

$$+ \left(- \frac{308\,385}{16} e^{12} b^7 c^2 + \frac{1867\,635}{32} e^{10} b^5 c^4 + \frac{93\,555}{32} e^6 b c^8 \right.$$

$$+ \frac{1155}{2} e^{14} b^9 - \frac{4163\,775}{128} e^8 b^3 c^6 \Bigg) l_6(\psi)$$

$$+ \left(- \frac{5445}{2} e^{14} b^8 c + \frac{308\,385}{16} e^{12} b^6 c^3 - \frac{696\,465}{32} e^{10} b^4 c^5 \right.$$

$$+ \frac{620\,235}{128} e^8 b^2 c^7 - \frac{3465}{32} c^9 e^6 \Bigg) l_7(\psi)$$

$$+ \left(- \frac{495}{4} e^{16} b^9 + \frac{5445}{2} e^{14} b^7 c^2 - \frac{197\,505}{32} e^{12} b^5 c^4 \right.$$

$$+ \frac{93\,555}{32} e^{10} b^3 c^6 - \frac{121\,275}{512} e^8 b c^8 \Bigg) l_8(\psi)$$

$$+ \left(\frac{495}{4} e^{16} b^8 c - \frac{1155}{2} e^{14} b^6 c^3 + \frac{17\,325}{32} e^{12} b^4 c^5 \right.$$

$$- \frac{3465}{32} e^{10} b^2 c^7 + \frac{1155}{512} c^9 e^8 \Bigg) l_9(\psi) \Bigg] \Bigg\},$$

$$c = \lambda + \cos\psi, \quad b = \sin\psi, \quad e = \sqrt{1 - \lambda^2}.$$

APPENDIX II

$$M_{2;20}(\psi) = \frac{537\,251}{4\,294\,967\,296} \Bigg\{ 988 \Bigg[\left(- \frac{17\,955}{2} e^2 c^{17} b^2 \right.$$

$$+ 101745 e^4 b^4 c^{15} - \frac{3561075}{8} e^6 b^6 c^{13}$$

$$+ \frac{27\,776\,385}{32} e^8 b^8 c^{11} - \frac{101\,846\,745}{128} e^{10} b^{10} c^9$$

$$+ \frac{43\,648\,605}{128} e^{12} b^{12} c^7 - \frac{130\,945\,815}{2048} e^{14} b^{14} c^5$$

$$+ \frac{72\,747\,675}{16384} e^{16} b^{16} c^3 - \frac{4849\,845}{65536} e^{18} b^{18} c + 210 c^{19} \Bigg) l_0(\psi)$$

$$+ \left(- \frac{1119\,195}{2} e^4 c^{16} b^3 + 21945 e^2 c^{18} b \right.$$

$$+ \frac{16\,787\,925}{4} e^6 b^5 c^{14} - \frac{101\,846\,745}{8} e^8 b^7 c^{12}$$

$$+ \frac{1\,120\,314\,195}{64} e^{10} b^9 c^{10} - \frac{1\,440\,403\,965}{128} e^{12} b^{11} c^8$$

$$+ \frac{3\,360\,942\,585}{1024} e^{14} b^{13} c^6 - \frac{800\,224\,425}{2048} e^{16} b^{15} c^4$$

$$+ \frac{480\,134\,655}{32768} e^{18} b^{17} c^2 - \frac{4\,849\,845}{65536} e^{20} b^{19} \Bigg) l_1(\psi)$$

$$+ \Bigg(-\frac{112\,021\,245}{8} e^6 c^{15} b^4 + 951615\, e^4 c^{17} b^2$$

$$- \frac{17\,955}{2} e^2 c^{19} + \frac{557\,664\,345}{8} e^8 b^6 c^{13}$$

$$+ \frac{480\,134\,655}{32768} e^{20} b^{18} c - \frac{56\,612\,240\,685}{65536} e^{18} b^{16} c^3$$

$$+ \frac{24\,923\,353\,455}{2048} e^{16} b^{14} c^5 - \frac{129\,243\,519\,405}{2048} e^{14} b^{12} c^7$$

$$+ \frac{567\,431\,865}{4} e^{12} b^{10} c^9 - \frac{18\,804\,612\,645}{128} e^{10} b^8 c^{11} \Bigg) l_2(\psi)$$

$$+ \Bigg(-\frac{1\,473\,979\,815}{8} e^8 c^{14} b^5 + \frac{41\,410\,215}{2} e^6 c^{16} b^3$$

$$- \frac{1\,119\,195}{2} e^4 c^{18} b + \frac{9\,879\,134\,265}{16} e^{10} b^7 c^{12}$$

$$- \frac{56\,612\,240\,685}{65536} e^{20} b^{17} c^2 + \frac{92\,345\,898\,645}{4096} e^{18} b^{15} c^4$$

$$- \frac{377\,545\,883\,715}{2048} e^{16} b^{13} c^6 + \frac{312\,567\,660\,405}{512} e^{14} b^{11} c^8$$

$$- \frac{116\,032\,541\,625}{128} e^{12} b^9 c^{10} + \frac{72\,747\,675}{16384} e^{22} b^{19} \Bigg) l_3(\psi)$$

$$+ \Bigg(-\frac{91\,464\,786\,945}{64} e^{10} c^{13} b^6 + \frac{4\,057\,895\,835}{16} e^8 c^{15} b^4$$

$$- \frac{112\,021\,245}{8} e^6 c^{17} b^2 + 101745\, e^4 c^{19}$$

$$+ \frac{418\,252\,837\,275}{128} e^{12} b^8 c^{11} - \frac{800\,224\,425}{2048} e^{22} b^{18} c$$

$$+ \frac{92\,345\,898\,645}{4096} e^{20} b^{16} c^3 - \frac{5\,068\,345\,066\,785}{16384} e^{18} b^{14} c^5$$

$$+ \frac{6\,356\,560\,895\,685}{4096} e^{16} b^{12} c^7$$

$$- \frac{6\,844\,814\,191\,215}{2048} e^{14} b^{10} c^9 \Bigg) l_4(\psi)$$

$$+ \Bigg(-\frac{445\,957\,797\,285}{64} e^{12} c^{12} b^7 + \frac{60\,359\,305\,545}{32} e^{10} c^{14} b^5$$

$$- \frac{1\,473\,979\,815}{8} e^8 c^{16} b^3 + \frac{11\,235\,631\,061\,655}{1024} e^{14} b^9 c^{10}$$

$$+ \frac{16\,787\,925}{4} e^6 c^{18} b + \frac{24\,923\,353\,455}{2048} e^{22} b^{17} c^2$$

$$- \frac{5\,068\,345\,066\,785}{16384} e^{20} b^{15} c^4$$

$$+ \frac{20\,061\,466\,289\,865}{8192} e^{18} b^{13} c^6$$

$$- \frac{15\,947\,672\,565\,825}{2048} e^{16} b^{11} c^8 - \frac{130\,945\,815}{2048} e^{24} b^{19} \Bigg) l_5(\psi)$$

$$+ \Bigg(-\frac{45\,518\,078\,720\,205}{2048} e^{14} c^{11} b^8$$

$$+ \frac{71\,662\,361\,085}{8} e^{12} c^{13} b^6 - \frac{91\,464\,786\,945}{64} e^{10} c^{15} b^4$$

$$- \frac{3\,561\,075}{8} e^6 c^{19} + \frac{49\,577\,467\,764\,825}{2048} e^{16} b^{10} c^9$$

$$+ \frac{557\,664\,345}{8} e^8 c^{17} b^2 + \frac{3\,360\,942\,585}{1024} e^{24} b^{18} c$$

$$- \frac{377\,545\,883\,715}{2048} e^{22} b^{16} c^3 + \frac{20\,061\,466\,289\,865}{8192} e^{20} b^{14} c^5$$

$$- \frac{193\,475\,919\,001\,365}{16384} e^{18} b^{12} c^7 \Bigg) l_6(\psi)$$

$$+ \Bigg(-\frac{97\,410\,839\,120\,595}{2048} e^{16} c^{10} b^9$$

$$+ \frac{7\,193\,712\,040\,515}{256} e^{14} c^{12} b^7 - \frac{445\,957\,797\,285}{64} e^{12} c^{14} b^6$$

$$+ \frac{146\,612\,477\,846\,835}{4096} e^{18} b^{11} c^8 + \frac{9\,879\,134\,265}{16} e^{10} c^{16} b^3$$

$$- \frac{129\,243\,519\,405}{2048} e^{24} b^{17} c^2 + \frac{6\,356\,560\,895\,685}{4096} e^{22} b^{15} c^4$$

$$- \frac{193\,475\,919\,001\,365}{16384} e^{20} b^{13} c^6$$

$$- \frac{101\,846\,745}{8} e^8 c^{18} b + \frac{43\,648\,605}{128} e^{26} b^{19} \Bigg) l_7(\psi)$$

$$+ \Bigg(-\frac{2\,271\,939\,876\,841\,635}{32768} e^{18} c^9 b^{10}$$

$$+ \frac{487\,779\,442\,306\,065}{8192} e^{16} c^{11} b^8$$

$$-\frac{45\,518\,078\,720\,205}{2048}e^{14}c^{13}b^6+\frac{27\,776\,385}{32}e^8c^{19}$$

$$+\frac{146\,612\,477\,846\,835}{4096}e^{20}b^{12}c^7$$

$$+\frac{418\,252\,837\,275}{128}e^{12}c^{15}b^4-\frac{1\,440\,403\,965}{128}e^{26}b^{18}c$$

$$+\frac{312\,567\,660\,405}{512}e^{24}b^{16}c^3$$

$$-\frac{15\,947\,672\,565\,825}{2048}e^{22}b^{14}c^5$$

$$-\frac{18\,804\,612\,645}{128}e^{10}c^{17}b^2\Big)l_8(\psi)$$

$$+\Big(-\frac{2\,271\,939\,876\,841\,635}{32\,768}e^{20}c^8b^{11}$$

$$+\frac{1\,415\,489\,617\,707\,165}{16\,384}e^{18}c^{10}b^9$$

$$-\frac{97\,410\,839\,120\,595}{2048}e^{16}c^{12}b^7-\frac{101\,846\,745}{128}e^{28}b^{19}$$

$$+\frac{11\,235\,631\,061\,655}{1024}e^{14}c^{14}b^5$$

$$+\frac{49\,577\,467\,764\,825}{2048}e^{22}b^{13}c^6+\frac{567\,431\,865}{4}e^{26}b^{17}c^2$$

$$+\frac{1\,120\,314\,195}{64}e^{10}c^{18}b-\frac{6\,844\,814\,191\,215}{2048}e^{24}b^{15}c^4$$

$$-\frac{116\,032\,541\,625}{128}e^{12}c^{16}b^3\Big)l_9(\psi)$$

$$+\Big(-\frac{97\,410\,839\,120\,595}{2048}e^{22}c^7b^{12}$$

$$+\frac{1\,415\,489\,617\,707\,165}{16\,384}e^{20}c^9b^{10}$$

$$-\frac{2\,271\,939\,876\,841\,635}{32\,768}e^{18}c^{11}b^8-\frac{101\,846\,745}{128}e^{10}c^{19}$$

$$+\frac{11\,235\,631\,061\,655}{1024}e^{24}b^{14}c^5$$

$$+\frac{49\,577\,467\,764\,825}{2048}e^{16}c^{13}b^6$$

$$+\frac{1\,120\,314\,195}{64}e^{28}b^{18}c+\frac{567\,431\,865}{4}e^{12}c^{17}b^2$$

$$-\frac{116\,032\,541\,625}{128}e^{26}b^{16}c^3$$

$$-\frac{6\,844\,814\,191\,215}{2048}e^{14}c^{15}b^4\Big)l_{10}(\psi)$$

$$+\Big(-\frac{45\,518\,078\,720\,205}{2048}e^{24}c^6b^{13}$$

$$+\frac{487\,779\,442\,306\,065}{8192}e^{22}c^8b^{11}$$

$$+\frac{27\,776\,385}{32}e^{30}b^{19}-\frac{2\,271\,939\,876\,841\,635}{32\,768}e^{20}c^{10}b^9$$

$$+\frac{418\,252\,837\,275}{128}e^{26}b^{15}c^4$$

$$+\frac{146\,612\,477\,846\,835}{4096}e^{18}c^{12}b^7-\frac{1\,440\,403\,965}{128}e^{12}c^{18}b$$

$$+\frac{312\,567\,660\,405}{512}e^{14}c^{16}b^3-\frac{18\,804\,612\,645}{128}e^{28}b^{17}c^2$$

$$-\frac{15\,947\,672\,565\,825}{2048}e^{16}c^{14}b^5\Big)l_{11}(\psi)$$

$$+\Big(-\frac{445\,957\,797\,285}{64}e^{26}c^5b^{14}$$

$$+\frac{7\,193\,712\,040\,515}{256}e^{24}c^7b^{12}$$

$$-\frac{97\,410\,839\,120\,595}{2048}e^{22}c^9b^{1c}+\frac{9\,879\,134\,265}{16}e^{28}b^{16}c^3$$

$$+\frac{146\,612\,477\,846\,835}{4096}e^{20}c^{11}b^8-\frac{101\,846\,745}{8}e^{30}b^{18}c$$

$$-\frac{129\,243\,519\,405}{2048}e^{14}c^{17}b^2+\frac{6\,356\,560\,895\,685}{4096}e^{16}c^{15}b^4$$

$$-\frac{193\,475\,919\,001\,365}{16\,384}e^{18}c^{13}b^6$$

$$+\frac{43\,648\,605}{128}e^{12}c^{19}\Big)l_{12}(\psi)$$

$$+\Big(-\frac{91\,464\,786\,945}{64}e^{28}c^4b^{15}+\frac{71\,662\,361\,085}{8}e^{26}c^6b^{13}$$

$$-\frac{45\,518\,078\,720\,205}{2048}e^{24}c^8b^{11}-\frac{3\,561\,075}{8}e^{32}b^{19}$$

$$+\frac{557\,664\,345}{8}e^{30}b^{17}c^2+\frac{49\,577\,467\,764\,825}{2048}e^{22}c^{10}b^9$$

$$-\frac{193\,475\,919\,001\,365}{16\,384}e^{20}c^{12}b^7+\frac{3\,360\,942\,585}{1024}e^{14}c^{18}b$$

$$-\frac{377\,545\,883\,715}{2048}e^{16}c^{16}b^3$$

$$+ \frac{20\,061\,466\,289\,865}{8192} e^{18} c^{14} b^5 \Bigg) l_{13}(\psi)$$

$$+ \Bigg(-\frac{1\,473\,979\,815}{8} e^{30} c^3 b^{16} + \frac{60\,359\,305\,545}{32} e^{28} c^5 b^{14}$$

$$- \frac{445\,957\,797\,285}{64} e^{26} c^7 b^{12} + \frac{16\,787\,925}{4} e^{32} b^{18} c$$

$$+ \frac{11\,235\,631\,061\,655}{1024} e^{24} c^9 b^{10}$$

$$- \frac{15\,947\,672\,565\,825}{2048} e^{22} c^{11} b^8$$

$$+ \frac{24\,923\,353\,455}{2048} e^{16} c^{17} b^2 - \frac{5\,068\,345\,066\,785}{16\,384} e^{18} c^{15} b^4$$

$$+ \frac{20\,061\,466\,289\,865}{8192} e^{20} c^{13} b^6$$

$$- \frac{130\,945\,815}{2048} e^{14} c^{19} \Bigg) l_{14}(\psi)$$

$$+ \Bigg(-\frac{112\,021\,245}{8} e^{32} c^2 b^{17} + \frac{4\,057\,895\,835}{16} e^{30} c^4 b^{15}$$

$$- \frac{91\,464\,786\,945}{64} e^{28} c^6 b^{13} + 101\,745 e^{34} b^{19}$$

$$+ \frac{418\,252\,837\,275}{128} e^{26} c^8 b^{11} - \frac{6\,844\,814\,191\,215}{2048} e^{24} c^{10} b^9$$

$$- \frac{800\,224\,425}{2048} e^{16} c^{18} b + \frac{92\,345\,898\,645}{4096} e^{18} c^{16} b^3$$

$$- \frac{5\,068\,345\,066\,785}{16\,384} e^{20} c^{14} b^5$$

$$+ \frac{6\,356\,560\,895\,685}{4096} e^{22} c^{12} b^7 \Bigg) l_{15}(\psi)$$

$$+ \Bigg(-\frac{1\,119\,195}{2} e^{34} c b^{18} + \frac{41\,410\,215}{2} e^{32} c^3 b^{16}$$

$$- \frac{1\,473\,979\,815}{8} e^{30} c^5 b^{14} + \frac{9\,879\,134\,265}{16} e^{28} c^7 b^{12}$$

$$- \frac{116\,032\,541\,625}{128} e^{26} c^9 b^{10} - \frac{56\,612\,240\,685}{65\,536} e^{18} c^{17} b^2$$

$$+ \frac{92\,345\,898\,645}{4096} e^{20} c^{15} b^4 - \frac{377\,545\,883\,715}{2048} e^{22} c^{13} b^6$$

$$+ \frac{312\,567\,660\,405}{512} e^{24} c^{11} b^8 + \frac{72\,747\,675}{16\,384} e^{16} c^{19} \Bigg) l_{16}(\psi)$$

$$+ \Bigg(951\,615 e^{34} c^2 b^{17} - \frac{112\,021\,245}{8} e^{32} c^4 b^{15}$$

$$- \frac{17\,955}{2} b^{19} e^{36} + \frac{557\,664\,345}{8} e^{30} c^6 b^{13}$$

$$- \frac{18\,804\,612\,645}{128} e^{28} c^8 b^{11} + \frac{480\,134\,655}{32\,768} e^{18} c^{18} b$$

$$- \frac{56\,612\,240\,685}{65\,536} e^{20} c^{16} b^3 + \frac{24\,923\,353\,455}{2048} e^{22} c^{14} b^5$$

$$- \frac{129\,243\,519\,405}{2048} e^{24} c^{12} b^7$$

$$+ \frac{567\,431\,865}{4} e^{26} c^{10} b^9 \Bigg) l_{17}(\psi)$$

$$+ \Bigg(21\,945 c e^{36} b^{18} - \frac{1\,119\,195}{2} e^{34} c^3 b^{16}$$

$$+ \frac{16\,787\,925}{4} e^{32} c^5 b^{14} - \frac{101\,846\,745}{8} e^{30} c^7 b^{12}$$

$$+ \frac{1\,120\,314\,195}{64} e^{28} c^9 b^{10} - \frac{1\,440\,403\,965}{128} e^{26} c^{11} b^8$$

$$+ \frac{3\,360\,942\,585}{1024} e^{24} c^{13} b^6 - \frac{800\,224\,425}{2048} e^{22} c^{15} b^4$$

$$+ \frac{480\,134\,655}{32\,768} e^{20} c^{17} b^2 - \frac{4\,849\,845}{65\,536} e^{18} c^{19} \Bigg) l_{18}(\psi)$$

$$+ \Bigg(-\frac{17\,955}{2} e^{36} b^{17} c^2 + 101\,745 e^{34} b^{15} c^4$$

$$- \frac{3\,561\,075}{8} e^{32} b^{13} c^6 + \frac{27\,776\,385}{32} e^{30} b^{11} c^8$$

$$- \frac{101\,846\,745}{128} e^{28} b^9 c^{10} + \frac{43\,648\,605}{128} e^{26} b^7 c^{12}$$

$$- \frac{130\,945\,815}{2048} e^{24} b^5 c^{14} + \frac{72\,747\,675}{16\,384} e^{22} b^3 c^{16}$$

$$- \frac{4\,849\,845}{65\,536} e^{20} b c^{18} + 210 e^{38} b^{19} \Bigg) l_{19}(\psi) \Bigg]$$

$$+ 415 i e \Omega \Bigg[\Bigg(\frac{101\,745}{4} e^2 b^3 c^{16} - \frac{1\,526\,175}{8} e^4 b^5 c^{14}$$

$$+ \frac{9\,258\,795}{16} e^6 b^7 c^{12} - \frac{101\,846\,745}{128} e^8 b^9 c^{10}$$

$$+ \frac{130\,945\,815}{256} e^{10} b^{11} c^8 - \frac{305\,540\,235}{2048} e^{12} b^{13} c^6$$

$$+ \frac{72\,747\,675}{4096} e^{14} b^{15} c^4 - \frac{43\,648\,605}{65\,536} e^{16} b^{17} c^2$$

$$+ \frac{440\,895}{131\,072} e^{18} b^{19} - \frac{1995}{2} bc^{18} \Bigg) l_0(\psi)$$

$$+ \Bigg(\frac{10\,886\,715}{8} e^4 b^4 c^{15} - \frac{377\,055}{4} e^2 c^{17} b^2$$

$$- \frac{107\,544\,465}{16} e^6 b^6 c^{13} + \frac{1\,805\,465\,025}{128} e^8 b^8 c^{11}$$

$$- \frac{3\,477\,338\,865}{256} e^{10} b^{10} c^9 + \frac{12\,352\,555\,215}{2048} e^{12} b^{12} c^7$$

$$- \frac{4\,757\,697\,945}{4096} e^{14} b^{14} c^5 + \frac{1995}{2} c^{19}$$

$$+ \frac{5\,397\,877\,485}{65\,536} e^{16} b^{16} c^3 - \frac{182\,971\,425}{131\,072} e^{18} b^{18} c \Bigg) l_1(\psi)$$

$$+ \Bigg(-\frac{3\,027\,625\,965}{32} e^8 b^7 c^{12} + \frac{35\,311\,721\,445}{256} e^{10} b^9 c^{10}$$

$$+ \frac{377\,055}{4} e^2 c^{18} b - \frac{13\,125\,105}{4} e^4 c^{16} b^3$$

$$+ \frac{456\,936\,795}{16} e^6 b^5 c^{14} - \frac{94\,673\,824\,245}{1024} e^{12} b^{11} c^8$$

$$- \frac{43\,648\,605}{65\,536} e^{20} b^{19} + \frac{17\,011\,051\,785}{131\,072} e^{18} b^{17} c^2$$

$$- \frac{27\,804\,161\,385}{8192} e^{16} b^{15} c^4$$

$$+ \frac{113\,966\,507\,655}{4096} e^{14} b^{13} c^6 \Bigg) l_2(\psi)$$

$$+ \Bigg(-\frac{181\,153\,614\,915}{256} e^{10} b^8 c^{11}$$

$$+ \frac{734\,504\,175\,405}{1024} e^{12} b^{10} c^9 - \frac{1\,355\,478\,329\,205}{4096} e^{14} b^{12} c^7$$

$$+ \frac{13\,125\,105}{4} e^4 c^{17} b^2 - \frac{910\,922\,985}{16} e^6 c^{15} b^4$$

$$+ \frac{10\,038\,670\,425}{32} e^8 b^6 c^{13} + \frac{5\,397\,877\,485}{65\,536} e^{20} b^{18} c$$

$$- \frac{624\,776\,873\,175}{131\,072} e^{18} b^{16} c^3$$

$$+ \frac{537\,794\,462\,205}{8192} e^{16} b^{14} c^5 - \frac{101\,745}{4} e^2 c^{19} \Bigg) l_3(\psi)$$

$$+ \Bigg(-\frac{6\,457\,301\,876\,025}{2048} e^{12} b^9 c^{10}$$

$$+ \frac{9\,067\,517\,554\,095}{4096} e^{14} b^{11} c^8$$

$$- \frac{11\,315\,478\,909\,735}{16\,384} e^{16} b^{13} c^6 - \frac{10\,886\,715}{8} e^4 c^{18} b$$

$$+ \frac{910\,922\,985}{16} e^6 c^{16} b^3 - \frac{36\,011\,930\,535}{64} e^8 c^{14} b^5$$

$$+ \frac{260\,186\,689\,035}{128} e^{10} b^7 c^{12} + \frac{72\,747\,675}{4096} e^{22} b^{19}$$

$$- \frac{27\,804\,161\,385}{8192} e^{20} b^{17} c^2$$

$$+ \frac{2\,841\,078\,440\,655}{32\,768} e^{18} b^{15} c^4 \Bigg) l_4(\psi)$$

$$+ \Bigg(-\frac{36\,376\,092\,677\,925}{4096} e^{14} b^{10} c^9$$

$$+ \frac{70\,102\,234\,897\,695}{16\,384} e^{16} b^{12} c^7$$

$$- \frac{28\,796\,615\,589\,105}{32\,768} e^{18} b^{14} c^5 - \frac{456\,936\,795}{16} e^6 c^{17} b^2$$

$$+ \frac{36\,011\,930\,535}{64} e^8 c^{15} b^4 - \frac{437\,433\,194\,205}{128} e^{10} c^{13} b^6$$

$$+ \frac{16\,976\,779\,693\,965}{2048} e^{12} b^8 c^{11} - \frac{4\,757\,697\,945}{4096} e^{22} b^{18} c$$

$$+ \frac{537\,794\,462\,205}{8192} e^{20} b^{16} c^3 + \frac{1\,526\,175}{8} e^4 c^{19} \Bigg) l_5(\psi)$$

$$+ \Bigg(\frac{107\,544\,465}{16} e^6 c^{18} b - \frac{133\,892\,925\,160\,995}{8192} e^{16} b^{11} c^8$$

$$+ \frac{174\,186\,589\,920\,525}{32\,768} e^{18} b^{13} c^6$$

$$- \frac{11\,315\,478\,909\,735}{16\,384} e^{20} b^{15} c^4 - \frac{10\,038\,670\,425}{32} e^8 c^{16} b^3$$

$$+ \frac{437\,433\,194\,205}{128} e^{10} c^{14} b^5 - \frac{6\,848\,147\,357\,415}{512} e^{12} c^{12} b^7$$

$$+ \frac{90\,588\,503\,960\,955}{4096} e^{14} b^9 c^{10}$$

$$+ \frac{113\,966\,507\,655}{4096} e^{22} b^{17} c^2 - \frac{305\,540\,235}{2048} e^{24} b^{19} \Bigg) l_6(\psi)$$

$$+ \Bigg(\frac{3\,027\,625\,965}{32} e^8 c^{17} b^2 - \frac{655\,162\,385\,283\,315}{32\,768} e^{18} b^{12} c^7$$

$$+ \frac{70\,102\,234\,897\,695}{16\,384} e^{20} b^{14} c^5$$

$$- \frac{1\,355\,478\,329\,205}{4096} e^{22} b^{16} c^3$$

$$+ \frac{323\,723\,094\,429\,735}{8192} e^{16} b^{10} c^9$$

$$- \frac{260\,186\,689\,035}{128} e^{10} c^{15} b^4 + \frac{6\,848\,147\,357\,415}{512} e^{12} c^{13} b^6$$

$$- \frac{142\,481\,264\,361\,345}{4096} e^{14} c^{11} b^8$$

$$+ \frac{12\,352\,555\,215}{2048} e^{24} b^{18} c - \frac{9\,258\,795}{16} e^6 c^{19} \Bigg) l_7(\psi)$$

$$+ \left(\frac{181\,153\,614\,915}{256} e^{10} c^{16} b^3 \right.$$

$$- \frac{133\,892\,925\,160\,995}{8192} e^{20} b^{13} c^6 - \frac{1\,805\,465\,025}{128} e^8 c^{18} b$$

$$+ \frac{9\,067\,517\,554\,095}{4096} e^{22} b^{15} c^4 - \frac{94\,673\,824\,245}{1024} e^{24} b^{17} c^2$$

$$- \frac{16\,976\,779\,693\,965}{2048} e^{12} c^{14} b^5$$

$$+ \frac{142\,481\,264\,361\,345}{4096} e^{14} c^{12} b^7$$

$$- \frac{2\,008\,258\,828\,631\,055}{32\,768} e^{16} c^{10} b^9$$

$$+ \frac{3\,138\,386\,978\,624\,625}{65\,536} e^{18} b^{11} c^8$$

$$+ \frac{130\,945\,815}{256} e^{26} b^{19} \Bigg) l_8(\psi)$$

$$+ \left(\frac{6\,457\,301\,876\,025}{2048} e^{12} c^{15} b^4 \right.$$

$$- \frac{36\,376\,092\,677\,925}{4096} e^{22} b^{14} c^5 - \frac{35\,311\,721\,445}{256} e^{10} c^{17} b^2$$

$$+ \frac{734\,504\,175\,405}{1024} e^{24} b^{16} c^3 - \frac{3\,477\,338\,865}{256} e^{26} b^{18} c$$

$$- \frac{90\,588\,503\,960\,955}{4096} e^{14} c^{13} b^6$$

$$+ \frac{2\,008\,258\,828\,631\,055}{32\,768} e^{16} c^{11} b^8$$

$$- \frac{4\,845\,557\,363\,581\,935}{65\,536} e^{18} c^9 b^{10}$$

$$+ \frac{323\,723\,094\,429\,735}{8192} e^{20} b^{12} c^7$$

$$+ \frac{101\,846\,745}{128} e^8 c^{19} \Bigg) l_9(\psi)$$

$$+ \left(\frac{36\,376\,092\,677\,925}{4096} e^{14} c^{14} b^5 \right.$$

$$- \frac{6\,457\,301\,876\,025}{2048} e^{24} b^{15} c^4 - \frac{734\,504\,175\,405}{1024} e^{12} c^{16} b^3$$

$$+ \frac{35\,311\,721\,445}{256} e^{26} b^{17} c^2 + \frac{3\,477\,338\,865}{256} e^{10} c^{18} b$$

$$- \frac{323\,723\,094\,429\,735}{8192} e^{16} c^{12} b^7$$

$$+ \frac{4\,845\,557\,363\,581\,935}{65\,536} e^{18} c^{10} b^9$$

$$- \frac{2\,008\,258\,828\,631\,055}{32\,768} e^{20} c^8 b^{11}$$

$$+ \frac{90\,588\,503\,960\,955}{4096} e^{22} b^{13} c^6$$

$$- \frac{101\,846\,745}{128} e^{28} b^{19} \Bigg) l_{10}(\psi)$$

$$+ \left(\frac{133\,892\,925\,160\,995}{8192} e^{16} c^{13} b^6 \right.$$

$$- \frac{181\,153\,614\,915}{256} e^{26} b^{16} c^3$$

$$- \frac{9\,067\,517\,554\,095}{4096} e^{14} c^{15} b^4 + \frac{1\,805\,465\,025}{128} e^{28} b^{18} c$$

$$+ \frac{94\,673\,824\,245}{1024} e^{12} c^{17} b^2$$

$$- \frac{3\,138\,386\,978\,624\,625}{65\,536} e^{18} c^{11} b^8$$

$$+ \frac{2\,008\,258\,828\,631\,055}{32\,768} e^{20} c^9 b^{10}$$

$$- \frac{142\,481\,264\,361\,345}{4096} e^{22} c^7 b^{12}$$

$$+ \frac{16\,976\,779\,693\,965}{2048} e^{24} b^{14} c^5$$

$$- \frac{130\,945\,815}{256} e^{10} c^{19} \Bigg) l_{11}(\psi)$$

$$+ \left(\frac{655\,162\,385\,283\,315}{32\,768} e^{18} c^{12} b^7 \right.$$

$$- \frac{3\,027\,625\,965}{32} e^{28} b^{17} c^2 - \frac{70\,102\,234\,897\,695}{16\,384} e^{16} c^{14} b^5$$

$$+ \frac{1\,355\,478\,329\,205}{4096} e^{14} c^{16} b^3$$

$$-\frac{323\,723\,094\,429\,735}{8192}e^{20}c^{10}b^9$$

$$+\frac{142\,481\,264\,361\,345}{4096}e^{22}c^8b^{11}$$

$$-\frac{6\,848\,147\,357\,415}{512}e^{24}c^6b^{13}+\frac{260\,186\,689\,035}{128}e^{26}b^{15}c^4$$

$$-\frac{12\,352\,555\,215}{2048}e^{12}c^{18}b+\frac{9\,258\,795}{16}e^{30}b^{19}\Bigg)l_{12}(\psi)$$

$$+\Bigg(\frac{133\,892\,925\,160\,995}{8192}e^{20}c^{11}b^8-\frac{107\,544\,465}{16}e^{30}b^{18}c$$

$$-\frac{174\,186\,589\,920\,525}{32\,768}e^{18}c^{13}b^6$$

$$+\frac{11\,315\,478\,909\,735}{16\,384}e^{16}c^{15}b^4$$

$$-\frac{90\,588\,503\,960\,955}{4096}e^{22}c^9b^{10}$$

$$+\frac{6\,848\,147\,357\,415}{512}e^{24}c^7b^{12}$$

$$-\frac{437\,433\,194\,205}{128}e^{26}c^5b^{14}+\frac{10\,038\,670\,425}{32}e^{28}b^{16}c^3$$

$$-\frac{113\,966\,507\,655}{4096}e^{14}c^{17}b^2+\frac{305\,540\,235}{2048}e^{12}c^{19}\Bigg)l_{13}(\psi)$$

$$+\Bigg(\frac{36\,376\,092\,677\,925}{4096}e^{22}c^{10}b^9$$

$$-\frac{70\,102\,234\,897\,695}{16\,384}e^{20}c^{12}b^7$$

$$+\frac{28\,796\,615\,589\,105}{32\,768}e^{18}c^{14}b^5$$

$$-\frac{16\,976\,779\,693\,965}{2048}e^{24}c^8b^{11}$$

$$+\frac{437\,433\,194\,205}{128}e^{26}c^6b^{13}$$

$$-\frac{36\,011\,930\,535}{64}e^{28}c^4b^{15}+\frac{456\,936\,795}{16}e^{30}b^{17}c^2$$

$$+\frac{4\,757\,697\,945}{4096}e^{14}c^{18}b$$

$$-\frac{537\,794\,462\,205}{8192}e^{16}c^{16}b^3-\frac{1\,526\,175}{8}e^{32}b^{19}\Bigg)l_{14}(\psi)$$

$$+\Bigg(\frac{6\,457\,301\,876\,025}{2048}e^{24}c^9b^{10}$$

$$-\frac{9\,067\,517\,554\,095}{4096}e^{22}c^{11}b^8$$

$$+\frac{11\,315\,478\,909\,735}{16\,384}e^{20}c^{13}b^6+\frac{10\,886\,715}{8}e^{32}b^{18}c$$

$$-\frac{260\,186\,689\,035}{128}e^{26}c^7b^{12}+\frac{36\,011\,930\,535}{64}e^{28}c^5b^{14}$$

$$-\frac{910\,922\,985}{16}e^{30}c^3b^{16}-\frac{72\,747\,675}{4096}e^{14}c^{19}$$

$$+\frac{27\,804\,161\,385}{8192}e^{16}c^{17}b^2$$

$$-\frac{2\,841\,078\,440\,655}{32\,768}e^{18}c^{15}b^4\Bigg)l_{15}(\psi)$$

$$+\Bigg(\frac{181\,153\,614\,915}{256}e^{26}c^8b^{11}-\frac{734\,504\,175\,405}{1024}e^{24}c^{10}b^9$$

$$+\frac{1\,355\,478\,329\,205}{4096}e^{22}c^{12}b^7-\frac{10\,038\,670\,425}{32}e^{28}c^6b^{13}$$

$$+\frac{910\,922\,985}{16}e^{30}c^4b^{15}-\frac{13\,125\,105}{4}e^{32}c^2b^{17}$$

$$-\frac{5\,397\,877\,485}{65\,536}e^{16}c^{18}b+\frac{624\,776\,873\,175}{131\,072}e^{18}c^{16}b^3$$

$$-\frac{537\,794\,462\,205}{8192}e^{20}c^{14}b^5+\frac{101\,745}{4}e^{34}b^{19}\Bigg)l_{16}(\psi)$$

$$+\Bigg(\frac{3\,027\,625\,965}{32}e^{28}c^7b^{12}-\frac{35\,311\,721\,445}{256}e^{26}c^9b^{10}$$

$$+\frac{94\,673\,824\,245}{1024}e^{24}c^{11}b^8-\frac{456\,936\,795}{16}e^{30}c^5b^{14}$$

$$+\frac{13\,125\,105}{4}e^{32}c^3b^{16}-\frac{377\,055}{4}e^{34}cb^{18}$$

$$+\frac{43\,648\,605}{65\,536}e^{16}c^{19}-\frac{17\,011\,051\,785}{131\,072}e^{18}c^{17}b^2$$

$$+\frac{27\,804\,161\,385}{8192}e^{20}c^{15}b^4$$

$$-\frac{113\,966\,507\,655}{4096}e^{22}c^{13}b^6\Bigg)l_{17}(\psi)$$

$$+\Bigg(\frac{377\,055}{4}e^{34}c^2b^{17}-\frac{10\,886\,715}{8}e^{32}c^4b^{15}$$

$$+\frac{107\,544\,465}{16}e^{30}c^6b^{13}-\frac{1\,805\,465\,025}{128}e^{28}c^8b^{11}$$

$$+\frac{3\,477\,338\,865}{256}e^{26}c^{10}b^9-\frac{12\,352\,555\,215}{2048}e^{24}c^{12}b^7$$

$$+ \frac{4\,757\,697\,945}{4096} e^{22} c^{14} b^5 - \frac{5\,397\,877\,485}{65536} e^{20} c^{16} b^3$$

$$+ \frac{182\,971\,425}{131072} e^{18} c^{18} b - \frac{1995}{2} b^{19} e^{36} \bigg) l_{18}(\psi)$$

$$+ \bigg(-\frac{101\,745}{4} e^{34} c^3 b^{16} + \frac{1\,526\,175}{8} e^{32} c^5 b^{14}$$

$$- \frac{9\,258\,795}{16} e^{30} c^7 b^{12} + \frac{101\,846\,745}{128} e^{28} c^9 b^{10}$$

$$- \frac{130\,945\,815}{256} e^{26} c^{11} b^8 + \frac{305\,540\,235}{2048} e^{24} c^{13} b^6$$

$$- \frac{72\,747\,675}{4096} e^{22} c^{15} b^4 + \frac{43\,648\,605}{65536} e^{20} c^{17} b^2$$

$$- \frac{440\,895}{131072} e^{18} c^{19} + \frac{1995}{2} c e^{36} b^{18} \bigg) l_{19}(\psi) \bigg] \bigg\}.$$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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