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SOLVING DYNAMIC COUPLED THERMOELASTICITY PROBLEMS

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ABSTRACT

This paper presents a numerical method for solving a coupled dynamic thermoelasticity problem based on the finite element method. The formulation combines the implicit Newmark scheme for the mechanical part with the Crank-Nicolson scheme for the thermal field.

The study is restricted to a one-dimensional model describing the thermomechanical behavior of a homogeneous elastic rod. The rod is subjected to non-stationary temperature effects that generate transient thermal stresses and deformations. The governing equations of motion and heat conduction are written in coupled form and reduced to their integral representation. Spatial discretization is carried out using the finite element method with linear elements. For time integration, the Newmark and Crank–Nicolson schemes are employed to ensure stability and accuracy. Special attention is given to the consistent treatment of thermomechanical coupling terms during discretization. Numerical modeling is performed for a rod clamped at one end and subjected to exponential cooling at the free end. The computed distributions of displacement, strain, stress, and temperature are analyzed in detail. The obtained results demonstrate agreement with the theoretical principles of classical dynamic thermoelasticity. This confirms

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the reliability of the proposed algorithm for solving coupled transient thermomechanical problems.

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Key Words and Phrases: Thermoelasticity, Coupled thermal-mechanical analysis, FEM, Newmark method, Crank–Nicolson scheme

1 Introduction

This article presents a numerical method for solving a dynamic coupled thermoelasticity problem using the finite element method (FEM) along with the Newmark and Crank-Nicolson schemes, exemplified through a one-dimensional problem.

Coupled thermoelasticity problems involve the simultaneous interaction of thermal and mechanical fields in solids. These problems find applications in various fields such as mechanical engineering, aerospace engineering, structural mechanics, and the modeling of micro- and nanoscale devices. The dynamic formulation accounts for inertia, which is particularly crucial for fast-moving thermomechanical processes, pulsed thermal effects, and high-speed deformations. The theoretical foundations of classical thermoelasticity were established by A.K. Biot, who proposed a theory that incorporated thermal and mechanical vibrations in elastic bodies. Significant contributions were also made by S.P. Timoshenko, who developed engineering models for thermoelastic elements, and M. Hettinger, who focused on the theory of heat waves and the expansion of heat transfer equations [1, 2, 3].

Classical Biot's theory assumes instantaneous heat propagation; however, later approaches take into account a finite velocity of thermal propagation, which is important in high-frequency dynamic problems. Reference [4] provides a fundamental and systematic exposition of the FEM, detailing formulations of the variational principle, numerical schemes, and algorithms applicable to both stationary and non-stationary problems. It places particular emphasis on the development of stable discretizations for mechanics, heat conduction, and coupled thermomechanical problems.

The study referenced in [5] enhances the classical theory by providing a modern perspective on the algorithmic support of FEM, focusing on linear and dynamic analysis. The methods discussed, such as the Newmark time integration method, are widely utilized for solving dynamic problems. The foundational principles of thermomechanical models are largely drawn from the classical monograph [6], which offers a comprehensive theory of thermal stresses. This article specifically addresses elastic materials and non-stationary temperature fields.

A more contemporary perspective on heat transfer phenomena is presented in [7], which introduces a theory of thermal conductivity that incorporates two temperatures: macro-temperature and micro-temperature. This approach accounts for internal microstructural effects that cannot be described by the classical Fourier law. Article [8] focuses on the numerical modeling of thermomechanical problems in materials exhibiting a memory effect. The authors employ the FEM while considering time nonlinearity, which is particularly significant when modeling materials with elastic-plastic properties under non-stationary conditions.

By combining classical and modern theories, a comprehensive model of the coupled thermomechanical problem can be developed, capable of factoring in thermal stresses and the complex behavior of materials under dynamic conditions. The article details a numerical method for solving the coupled dynamic problem of thermoelasticity using the finite element method, the implicit Newmark scheme, and the Crank-Nicolson scheme. Numerical modeling is performed on a rod that is fixed at one end and subjected to exponential cooling at the free end. The analysis of the results obtained demonstrates that the behavior of the temperature and strain and stress fields aligns with the principles of classical dynamic thermoelasticity

2 Problem statement and solution method

The governing equations in the absence of heat release are as follows:

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\rho\frac{\partial^2 u}{\partial t^2}-(3\lambda+2\mu)\alpha\frac{\partial T}{\partial x}=0, \qquad (1)$$

$$K\frac{\partial^2 T}{\partial x^2}-\rho c\frac{\partial T}{\partial t}-(3\lambda+2\mu)\alpha T_0\frac{\partial^2 u}{\partial x\partial t}=0, \qquad (2)$$

where λ and μ - are the Lame constants, and ρ, K and c are the mass density, thermal conductivity, and specific heat capacity of the rod.

Initial conditions of the rod are:

$$u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0,\ T(x,0)=T_0 \qquad (3)$$

and boundary conditions are:

$$u(0,t)=0,\ \sigma_x(L,t)=0,\ T(0,t)=T_0,\ T(L,t)=T_0e^{-t/t_0}, \qquad (4)$$

where σ_{xx} – is the axial stress of the rod, defined as

$$\sigma_{xx}=(\lambda+2\mu)\frac{\partial u}{\partial x}-(3\lambda+2\mu)\alpha(T-T_0). \qquad \text{`}(5)$$

The system of coupled equations (1) and (2) has no general analytical solution. A finite element formulation can be developed based on the Galerkin method. The finite element model of the problem is obtained by discretizing the solution domain into a number of arbitrary elements. In each basic element (e), the components of displacement and temperature change are approximated by functions of the form $w(x)$:

$$u(x,t)^{(e)}=U_{mi}(t)N_m(x)=N_iU_i+N_jU_j=\langle N\rangle^{(e)}\{U\}^{(e)} \qquad (6)$$

where the piecewise linear shape function $<N>$ is $N_i=(L-\eta)/L$ and $\eta=x-x_i$.

Similarly, the temperature change $v(x)$ is approximated by

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$$T(x,t)^{(e)} = T_m(t)N_m(x) = N_iT_i + N_jT_j = \langle N\rangle^{(e)}\{T\}^{(e)}. \tag{7}$$

The integral formulation of the equation of motion (1) is multiplied by a function of form $w(x)$ and integrated over Ω:

$$\int_\Omega \left[(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial x}\right] w(x)dx = 0. \tag{8}$$

We integrate the first derivative by parts:

$$-\int_\Omega (\lambda+2\mu)\frac{\partial u}{\partial x}\frac{dw}{dx}dx + \int_\Omega \rho\frac{\partial^2 u}{\partial t^2}wdx + \int_\Omega (3\lambda+2\mu)\alpha\frac{\partial T}{\partial x}wdx = 0. \tag{9}$$

The integral formulation of the heat equation (2) is multiplied by a function of the form $v(x)$ and integrated over Ω:

$$\int_\Omega \left[K\frac{\partial^2 T}{\partial x^2} - \rho c\frac{\partial T}{\partial t} - (3\lambda+2\mu)\alpha T_0\frac{\partial^2 u}{\partial x\partial t}\right] v(x)dx = 0. \tag{10}$$

We integrate the first term by parts:

$$-\int_\Omega K\frac{\partial T}{\partial x}\frac{dv}{dx}dx + \int_\Omega \rho c\frac{\partial T}{\partial x}v\,dx + \int_\Omega (3\lambda+2\mu)\alpha T_0\frac{\partial u}{\partial x}\frac{dv}{dt}dx = 0. \tag{11}$$

The problem is to determine the values of functions $u(x,t)\in V_u$ and $T(x,t)\in V_T$ for all $w(x)\in V_u,\ v(x)\in V_T$.

After spatial discretization by the FEM, we have a system of ordinary first-order differential equations (12) for the temperature $T(t)$ and of second-order equations (13) for the displacements $u(t)$:

$$M_T\dot{T} + K_T T = C_{Tu}\dot{u} + f_T(t), \tag{12}$$

$$M_u\ddot{u} + K_u u = C_{uT}T + f_u(t), \tag{13}$$

which are formed by summation over finite elements (e):

$$K_T^{(e)} = \int_{\Omega^{(e)}} K\frac{\partial T}{\partial x}\frac{\partial v}{\partial x}dx = K\frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix};$$

$$M_T^{(e)} = \int_{\Omega^{(e)}} \rho c\frac{\partial T}{\partial x}vdx = \rho c\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix};$$

$$C_{Tu}^{(e)} = \int\limits_{\Omega^{(e)}} (3\lambda+2\mu)\alpha T_0 \frac{\partial u}{\partial x}\frac{\partial v}{\partial t}dx = \\ = (3\lambda+2\mu)\alpha T_0 \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix};$$

$$K_u^{(e)} = \int\limits_{\Omega^{(e)}} (\lambda+2\mu)\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}dx = (\lambda+2\mu)\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

$$M_u^{(e)} = \int\limits_{\Omega^{(e)}} \rho\frac{\partial^2 u}{\partial t_2}wdx = \rho\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix};$$

$$C_{uT}^{(e)} = \int\limits_{\Omega^{(e)}} (3\lambda+2\mu)\alpha\frac{\partial T}{\partial x}wdx = (3\lambda+2\mu)\alpha\frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix};$$

$f_T(t)$ – is the external thermal effect on the system;

$f_u(t)$ – is the external load acting on the mechanical part of the system.

For numerical integration of the second-order equations (13), the Newmark method is used. Let u^n be displacements at step *n*, $\dot{u}^n$ – velocities, $\ddot{u}^n$ – accelerations, Δt – the time step; then, the Newmark formulas have the following form:

$$u^{n+1} = u^n + \Delta t\dot{u}^n + \frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{u}^n + 2\beta\ddot{u}^{n+1}\right], \tag{14}$$

$$\dot{u}^{n+1} = \dot{u}^n + \Delta t\left[(1-\gamma)\ddot{u}^n + \gamma\ddot{u}^{n+1}\right], \tag{15}$$

where $\beta = 1/4$, $\gamma = 1/2$ are taken for the stability of the implicit scheme.

Let us introduce the effective stiffness matrix:

$$K_{eff} = K_u + \frac{\gamma}{\beta\Delta t}M_u. \tag{16}$$

The right-hand side of (11) at step n+1 is:

$$R^{n+1} = C_{uT}T^{n+1} + f_u^{n+1} + M_u A, \tag{17}$$

where $A = \frac{1}{\beta\Delta t^2}u^n + \frac{1}{\beta\Delta t}\dot{u}^n + \left(\frac{1}{2\beta} - 1\right)\ddot{u}^n$.

To calculate the temperature of the first derivative in time (12), the Crank-Nicolson method is used:

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$$M_T \frac{T^{n+1} - T^n}{\Delta t} + K_T \frac{T^{n+1} + T^n}{2} = \\ = C_{Tu} \frac{\dot{u}^{n+1} + \dot{u}^n}{2} + \frac{f_T^{n+1} + f_T^n}{2}, \quad (18)$$

which is solved at each step with the Newmark equation (11):

$$\left(M_T + \frac{\Delta t}{2} K_T \right) T^{n+1} = \left(M_T - \frac{\Delta t}{2} K_T \right) T^n + \\ + \Delta t \left(C_{Tu} \frac{\dot{u}^{n+1} + \dot{u}^n}{2} + \frac{f_T^{n+1} + f_T^n}{2} \right) \quad (19)$$

Step-by-step implementation of the algorithm:

for each time step $n = 0, 1, 2, ..., t_2$ the following actions are performed:

a) based on the known initial values of u^n, $\dot{u}^n$, $\ddot{u}^n$, T^n the right-hand side of the temperature equation is formed and system (19) is solved for T^{n+1}.

b) the right-hand side of equation (13) with T^{n+1} is calculated and this system is solved for u^{n+1}.

c) the values of $\dot{u}^{n+1}$, $\ddot{u}^{n+1}$ are calculated using the Newmark formulas.

3 Solution to one-dimensional thermoelasticity problem

Consider a rod L long, thermally insulated along its entire length.

The initial temperature at $x = L$ abruptly increases by $T\left(L, t\right) = T_0 e^{-t/t_0}$, while the temperature on side $x = 0$ remains at $T_0 = 100^o C$. Input data to the problem are: total number of nodes $N = 32$; number of finite elements $Ne = N - 1;$;length of the rod $L = 0.1$m; time step $t = 0.01$ sec; number of time steps $t_1 = 6000$; initial temperature of the rod $T_0 = 100^o C$; constant of time change of temperature $t_0 = 0.1$; coefficient of thermal conductivity $k_{xx} = 60.0$W/(m·°C); density of the rod $\rho = 7800$kg/m3; heat capacity of the material $c = 500$ J/(kg·°C); modulus of elasticity $-E = 2.0 \cdot 10^{11}$ Pa; cross-sectional area of the rod $A = 0.0001$m²; coefficient of thermal expansion $\alpha = 1{,}2 \cdot 10^{-5}\ 1/\ ^\circ C$; Poisson's ratio – $\nu = 0.3$; Lame coefficients: $\lambda = E \cdot \nu / ((1+\nu)(1-2\nu))$; $\mu = E / (2 \cdot (1+\nu))$.

Determine the distribution of temperatures, displacements, and stresses in the rod, under fixed boundary conditions at $x=0$ and with free end $x=L$. Initially, the body is at rest and has a

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uniform temperature T_0.Since the rod is thermally insulated along its entire length, the temperature distribution changes only as a function of the length coordinate, and the problem is considered as a one-dimensional classical coupled thermoelasticity problem.

Next, we present an analysis of the results from numerical modeling of the thermomechanical behavior of a rod subjected to non-stationary temperature conditions. The problem analyzed involves one-sided exponential cooling at the free end of the rod, while the opposite boundary remains fixed, with the entire rod starting at a uniform initial temperature. The study focuses on the simultaneous changes in the temperature field, linear displacements, and normal stresses along the length of the rod over time.

Modeling the temperature and stress state of a homogeneous rod under non-stationary thermal conditions at one end represents a coupled dynamic problem of thermoelasticity. This formulation encompasses both thermal and mechanical aspects, which interact with one another, reflecting the fundamental principles of the theory of thermoelasticity.

4 Analysis of the calculation results

An analysis of the temperature distribution graph along the length of the rod at different time points (Fig. 1) reveals behavior that aligns with the principles of heat transfer.

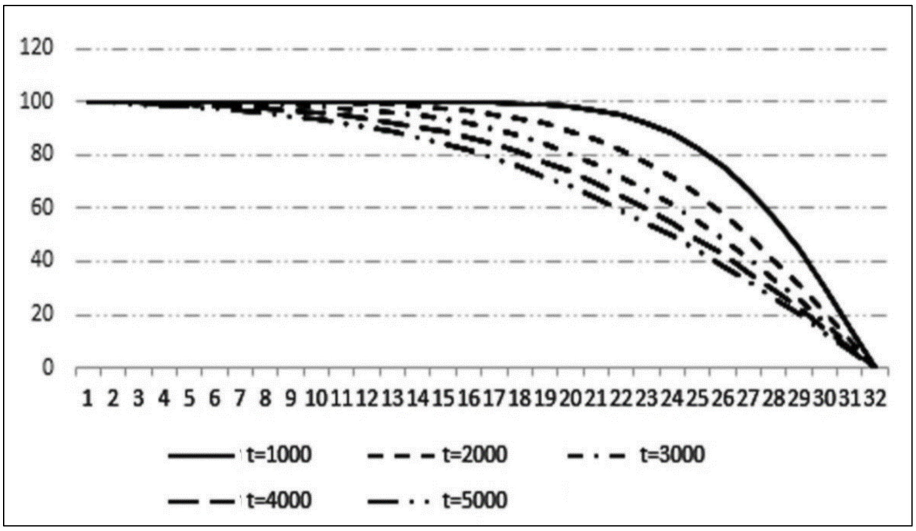

Figure 1. Temperature change in the rod by time steps

At the initial time points, the temperature along the entire rod is uniform and equal to T_0. As time progresses, the temperature at the free end of the rod decreases according to the formula $T(L,t) = T_0 e^{-t/t_0}$. This results in the formation of a temperature gradient from the fixed boundary on the left side to the free boundary on the right. Over time, the temperature of the right side of the rod declines rapidly, while the left side maintains its initial temperature for a longer period. This creates an uneven thermal field, which is essential for generating thermoelastic strains and stresses.

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The graph of the distribution of linear displacements (Fig. 2) indicates the presence of thermal expansion increasing from the fixed boundary ($x=0$) to the free end ($x=L$).

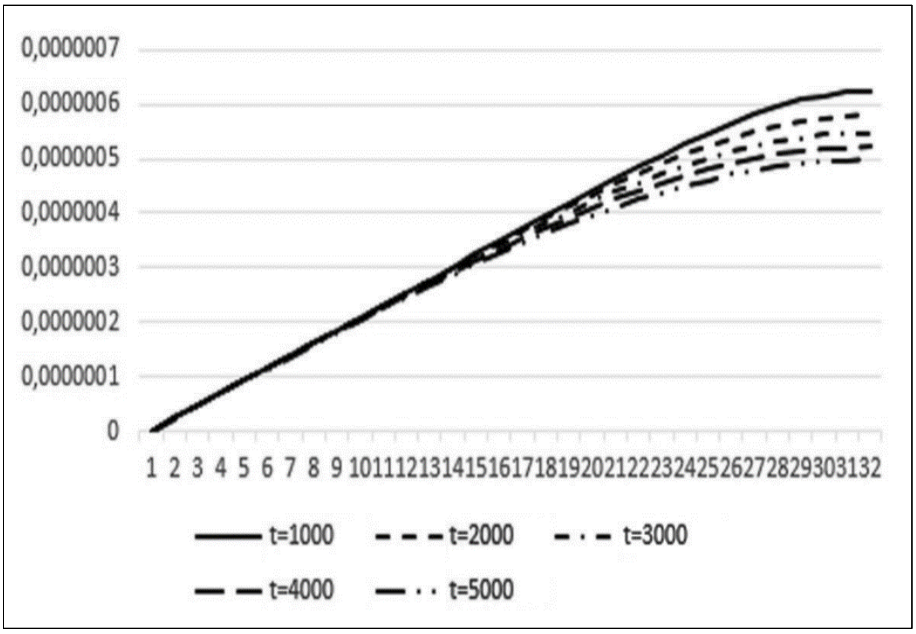

Figure 2. Change in displacements in the rod by time steps

At the initial time points, there are no displacements, but as the temperature gradient develops, the right section of the rod begins to shift to the right, demonstrating thermal elongation.

The highest displacement values occur at the free end and increase over time to a certain point. However, during the extended cooling period (t = 4000-5000), a decrease in temperature in the right section partially compensates for the thermal deformation, resulting in a reduction of displacements. This observed pattern of displacement changes confirms the reversible (elastic) nature of the deformations and their dependence on the thermal field.

The distribution of normal stresses σ_{xx} along the length of the rod, illustrated in Figure 3, reveals significant compressive stresses in the left section of the structure. This phenomenon occurs because free thermal expansion is restricted in the fixation zone. The maximum stress values (in terms of magnitude) are observed at intermediate time points (t = 1000-2000), when the temperature gradient between the ends of the rod is most pronounced. As time progresses and the temperature difference along the rod diminishes, the stresses decrease. In contrast, the right section of the rod, which is free, exhibits stress values close to zero, aligning with the specified boundary conditions.

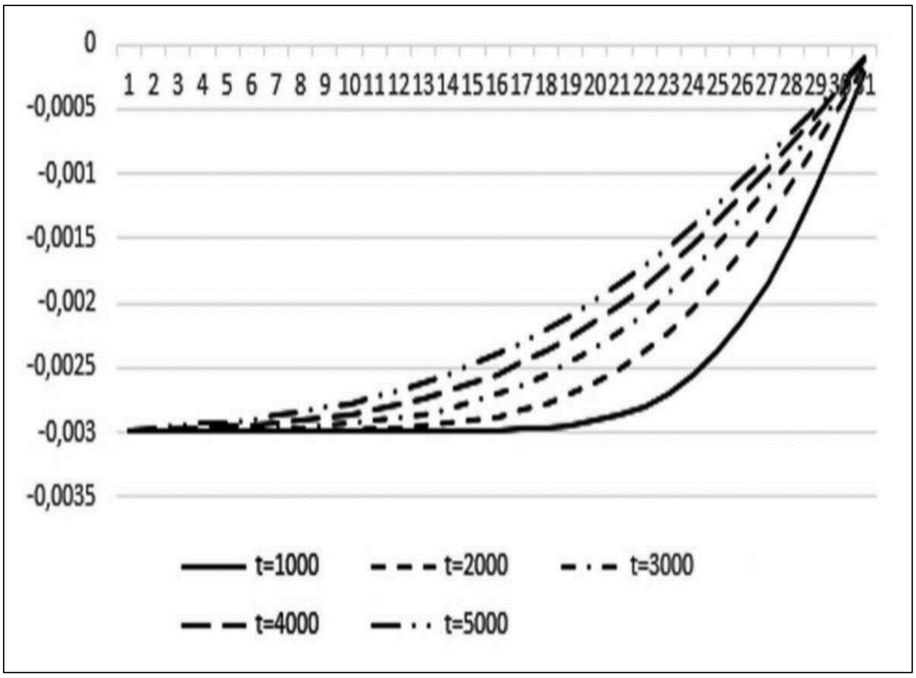

Figure 3. Change in axial stresses σ_{xx} in the rod by time steps

To evaluate the thermomechanical response of the structure, a numerical analysis of temperature, displacement, and stress evolution over time was performed. Special attention is given to point A, located at a distance of 0.05 m from the fixed end of the rod. Table 1 below presents the values of temperature T , displacement u, and axial stress σ_{xx} scaled by at several time steps. As the data show, the temperature at point A decreases gradually due to the transient thermal loading applied to the free end of the rod. This temperature drop is accompanied by a decrease in displacement, which indicates a reduction in thermal expansion as the thermal gradient weakens over time.

Table 1. Temperature, displacement, and stress over time

time	$T^{O}C$	$u \cdot 10^6$	$(\sigma_{xx} / E) \cdot 10^2$
t =1000	99.6487	0.3478	-0.2982
t =2000	95.5023	0.3458	-0.2875
t =3000	89.5025	0.3406	-0.2708
t =4000	83.7722	0.3335	- 0.2544
t =5000	78.7491	0.3257	- 0.2398

The axial stress (σ_{xx}) remains negative throughout the observation period, indicating compressive stress near the fixed boundary. The absolute value of the stress decreases over time, reflecting a stress relaxation process related to the alignment of the temperature distribution and the redistribution of the internal forces within the rod.

Overall, the behavior of temperature, displacement, and stress at point A is consistent with the expected thermoelastic response of the material. The system gradually transitions to a new quasi-equilibrium state, characterized by reduced thermal and mechanical gradients. This confirms the physical accuracy of the numerical model.

The results of numerical modeling accurately depict the physical behavior of thermomechanical interaction under non-stationary thermal loading. The temperature gradients formed lead to a corresponding redistribution of stresses, following the classical theory of thermoelasticity. Maximum stresses are concentrated in the area of rigid fixation, while thermal elongation occurs at the free end. Over time, the system moves toward a new equilibrium state, where the temperature field equalizes and stresses decrease.

5. Conclusion

The developed method effectively addresses the coupled problems of thermoelasticity under non-stationary temperature conditions. By using the finite element method in conjunction with the Newmark and Crank-Nicolson schemes, the approach ensures stability and a reliable approximation of the dynamics involved in thermomechanical processes. A numerical experiment was conducted on a rod fixed at one side end and exponential cooling at the free end, which demonstrated the expected distribution of temperature, strains, and stresses in accordance with the physical principles of the thermoelastic response. The results confirmed that deformations are reversible and are closely related to the temperature field. Maximum thermoelastic stresses were observed in the fixation zone, resulting from restricted thermal expansion. This observation underscores the importance of considering boundary conditions when analyzing the thermally stressed state of structures. Overall, the proposed approach can be applied to a wide range of problems related to dynamic thermomechanical effects in engineering practice.

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