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THE DENSITY AND THE LOCAL DENSITY OF SPACE OF THE PERMUTATION DEGREE AND HYPERSPACES

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ABSTRACT. In the work some are studied the density and the local density of space of the permutation degree and hyperspaces. Proved that for an infinite T_1-space X the followings: a) $d(X) = d(SP^nX)$; b) if $Y \subset X$ such that $ld(Y) = ld(X)$, then $ld(SP^nY) = ld(SP^nX)$. It also shown that the following: let X be an infinite T_1-space, n positive number, G_1 and G_2 subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then $d(X) = d(X^n) = d(SP^n_{G_1}X) = d(SP^n_{G_2}X) = d(SP^nX) = d(exp_nX)$.

1. INTRODUCTION

In the work [1] some are studied the weak density and the local weak density of space of the permutation degree and hyperspaces. In 2015 introduced the local density of topological spaces [2, 3]. In the work [2] proved that for stratifiable spaces the local density and the local weak density coincide, these cardinal numbers are preserved under open mappings, are inverse invariant of a class of closed irreducible mappings. In [4] proved that the density an infinite T_1-spaces is equally the density of the $N^{\varphi}_{\tau}-$nucleus of a space X. In our work proved that for an infinite T_1-space X the following: let X be an infinite T_1-space and Y is locally τ-dense in X. Then SP^nY is also locally τ-dense in SP^nX.

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2. Preliminaries

A set $A \subset X$ is dense in X if $\overline{A} = X$. The density is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X. This cardinal number is denoted by $d(X)$. If $d(X) \leq \aleph_0$, then we say that the space X is separable [5].

We say that the local density of a topological space X is τ at a point x, if τ is the smallest cardinal number such that x has a neighborhood of density τ in X. The local density at a point x is denoted by $ld(x)$. The local density of a topological space X is defined as the supremum of all numbers $ld(x)$ for $x \in X$: $ld(X) = \sup\{ld(x) : \ x \in X\}$ [2, 3]. It is known that, for any topological space we have $ld(X) \leq d(X)$.

Example 1. *Let us R real line with discrete topology. In the (R, τ_d) discrete topological space each point $x \in R$ has one-point neighborhood $\{x\}$. Then this implies that $ld(R, \tau_d) = 1$. On the other hand, in a discrete space the boundary set of any set is empty and so the only dense set is the space itself. This means that $d(R, \tau_d) = |R| = c$. Then we have that $1 = ld(R, \tau_d) < d(R, \tau_d) = c$.*

A permutation group X is the group of all permutations (i.s.one-one and onto mappings $X \to X$. A permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, 3, \ldots, n\}$, $S(X)$ is denoted by S_n, as well.

Let X^n be the n-th power of a compact X. The permutation group S_n of all permutations, acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient topology we denote by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus two points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are considered to be equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$.The space SP^nX is called the n -permutation degree of a space X. Equivalent relation by which we obtained space SP^nX is called the symmetric equivalence relation. The n-th permutation degree is always a quotient of X^n. Thus, the quotient map is denoted by as following: $\pi_n^s : X^n \to SP^nX$.

Where for every $x = (x_1, \ldots, x_n) \in X^n$, $\pi_n^s((x_1, \ldots, x_n)) = [(x_1, \ldots, x_n)]$ is an orbit of the point $x = (x_1, x_2, \ldots, x_n) \in X^n$.

The concept of a permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as group of permutations of coordinates. Consequently, it generates a G -symmetric equivalence relation on

X^n. This quotient space of the product of X^n under the G-symmetric equivalence relation is called G-permutation degree of the space X and it is denoted by SP^n_G. An operation $SP^n_G = SP^n$ is also the covariant functor in the category of compacts and it is said to be a functor of G -permutation degree. If $G = S_n$ then $SP^n_G = SP^n$. If the group G consists only of unique element then $SP^n_G = X^n$.

Let X be a T_1-space. The collection of all nonempty closed subsets of X we denote by $expX$. The family B of all sets in the form

$O\langle U_1, \ldots, U_n\rangle = \{F : F \in expX, F \subset \bigcup\limits_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, i = 1, 2, \ldots, n\}$, where $U_1, \ldots, U_n$ is a sequence of open sets of X, generates the topology on the set $expX$. This topology is called the Vietoris topology. The $expX$ with the Vietoris topology is called the exponential space or the hyperspace of X [6]. Let X be a T_1-space. Denote by exp_nX the set of all closed subsets of X cardinality of that is not greater than the cardinal number n, i.e. $exp_nX = \{F \in expX : |F| \leq n\}$.

Let's put $exp_\omega X = \cup\{exp_nX : n = 1, 2, \ldots\}$, $exp_cX = \{F \in expX : F$ is compact in $X\}$.

It is clear, that $exp_nX \subset exp_\omega X \subset exp_cX \subset expX$ for any topological space X. Moreover, if $G_1 \subset G_2$ for subgroups G_1, G_2 of the permutation group $\pi^s_n((x_1, x_2, \ldots, x_n)) = [x = (x_1, x_2, \ldots, x_n)] \in X^n$ then we get a sequence of the factorization of functors:

$X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to exp_nX$ [6].

Proposition 2.1. *[7] Let X be a space of local density τ and $f : X \to Y$ open continuous "onto" mapping. Then Y is space of local density τ.*

Proposition 2.2. *[8] $\pi^s_n : X^n \to SP^nX$ quotient map is an open, closed continuous onto mapping.*

Proposition 2.3. *[9] If X is a topological space, then $\exp_n X$ is dense in $\exp X$.*

Proposition 2.4. *[9] X is separable if and only if $\exp X$ is separable.*

Proposition 2.5. *[5] For every family of sets $\{A_s\}_{s\in S}$, where $A_s \subset X_s$, in the Cartesian product $\prod\limits_{s\in S} X_s$ we have $\overline{\prod\limits_{s\in S} A_s} = \prod\limits_{s\in S} \overline{A_s}$.*

Theorem 2.1. *(Hewitt-Marczewski-Pondiczery) [5] If $d(X_s) \leq \tau \geq \aleph_0$ for every $s \in S$ and $|S| \leq 2^\tau$, then $d\left(\prod\limits_{s\in S} X_s\right) \leq \tau$.*

Theorem 2.2. *[2] Let X, Y are T_1-spaces. If $f : X \to Y$ open map and $f(X) = Y$, then $ld(Y) \leq ld(X)$.*

3. Main Results

Theorem 3.1. *Let X be an infinite T_1-space. If Y is dense in Xtopological space, then SP^nY is also dense in SP^nX.*

Proof. We shall separate the proof of this theorem two parts. First we shall prove that if Y is dense in Xtopological space, then Y^n is also dense in X^n. Indeed if $Y$is dense in X, this means that $\overline{Y} = X$. By the proposition 2.5 we have $\overline{(Y^n)} = (\overline{Y})^n = X^n$. This shows that Y^n is dense in X^n. Now we shall prove that if Y^n is dense in X^n, then SP^nY is also dense in SP^nX. Let us Y^n be a dense subset of X^n and arbitrary $SP^nU \subset SP^nX$ open set. Since $\pi_n^s : X^n \to SP^nX$ map is continuous, the set $(\pi_n^s)^{-1}(SP^nU) \subset X^n$ is also open. Y^n is dense in X^n and so $(\pi_n^s)^{-1}(SP^nU) \cap Y^n \neq \emptyset$. Then there exists $y \in Y^n$ such that, $y \in (\pi_n^s)^{-1}(SP^nU)$. Then $\pi_n^s(y) \in SP^nU$ (and also $\pi_n^s(y) \in SP^nY$). This shows that for each open SP^nU set we have $SP^nU \cap SP^nY \neq \emptyset$. This means that SP^nY set is dense in SP^nX. Theorem 3.1 is proved. □

Proposition 3.1. *Let X be an infinite T_1-space, n positive number, G_1 and G_2 subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then*

$$d(X) = d(X^n) = d(SP^n_{G_1}X) = d(SP^n_{G_2}X) = d(SP^nX) = d(\exp_n X).$$

Proof. Let X is an infinite T_1-space. By $X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to \exp_n X$ and continuous mappings do not increase the density of topological spaces, it directly follows the inequalities

$$d(X) \geq d(X^n) \geq d(SP^n_{G_1}X) \geq d(SP^n_{G_2}X) \geq d(SP^nX) \geq d(\exp_n X)$$

and by propositions 2.3 and 2.4. $d(X) = d(\exp_n X)$. Hence, we obtain $d(X) = d(X^n) = d(SP^n_{G_1}X) = d(SP^n_{G_2}X) = d(SP^nX) = d(\exp_n X)$. Proposition 3.1 is proved. □

Corollary 3.1. *If X is an infinite T_1-space and Y subset of X such that $d(Y) = d(X)$ then $d(SP^nY) = d(SP^nX)$.*

Theorem 3.2. *If X topological space is locally τ-dense, then the product X^n is also locally τ-dense.*

Proof. Take an arbitrary point $x = (x_1, x_2, \ldots, x_n)$ from the product X^n. Since X is locally τ-dense, the $x_i \in X$ has a neighborhood U_i of density $\leq \tau$, for every $i = 1, 2, \ldots, n$. Since $d(X) \leq \tau$ by the Theorem 2.1 we have $d(\prod_{i=1}^{n} U_i) \leq \tau$. We know that $\prod_{i=1}^{n} U_i$ is a neighborhood of the point $x \in X^n$. This shows that we have found a τ-dense neighborhood of the point $x \in X^n$. The point x was chosen arbitrary, therefore the product X^n is locally τ-dense. Theorem 3.2 is proved. □

Theorem 3.3. *Let X be an infinite T_1-space and Y is a locally τ-dense set in X. Then SP^nY is also locally τ-dense in SP^nX.*

Proof. We shall separate the proof of the theorem two parts. First we can have easily from the Theorem 3.2 that if Y is a subset of X topological space such that, local density is τ, then local density of Y^n is also τ in the product X^n. Now, we shall prove that ifY^n is a locally τ-dense set in X^n, then SP^nY is also locally τ-dense in SP^nX. Indeed, since Y^n is locally τ-dense in X^n, for any point $y \in Y^n$ by the definition there exists a neighborhood $Oy \subset X^n$ such that Oy is τ-dense in X^n. Then that implies from the Theorem 3.2 that $SP^n(Oy)$ is also τ-dense in SP^nX. On the other hand, the quotient map $\pi_n^s : X^n \to SP^nX$ is an open map and so $SP^n(Oy)$ is a neighborhood of point $\pi_n^s(y) \in SP^nY$. Then we have that SP^nY is locally τ-dense in SP^nX. Theorem 3.3 is proved. □

Corollary 3.2. *If X is an infinite T_1-space and $Y \subset X$ such that $ld(Y) = ld(X)$, then $ld(SP^nY) = ld(SP^nX)$.*

Theorem 3.4. *For every X infinite T_1-space and every G subgroup of permutation group S_n, we have following equality $ld\,(X) = ld\,(SP_G^nX)$.*

Proof. First of all, we shall prove that $ld\,(SP_G^nX) \leq ld\,(X^n)$. Let us the following map $\pi_{n,G}^s : X^n \to SP_G^nX$,where $n \in N$. Clearly, the map $\pi_{n,G}^s$ is open. Then by the Theorem 2.2 we have $ld\,(SP_G^nX) \leq ld\,(X^n)$. Now we shall prove inverse inequality $ld\,(X^n) \leq ld\,(SP_G^nX)$. The mapping $\pi_{n,G}^s : X^n \to SP_G^nX$ is finitely multiple, because for each $y \in SP_G^nX$following relation holds: $\left|\left(\pi_{n,G}^s\right)^{-1}(y)\right| \leq n!$. Then the inequality $ld\,(X^n) \leq ld\,(SP_G^nX)$ is true and so $ld\,(X^n) = ld\,(SP_G^nX)$. Now we shall show that $ld\,(X^n) \leq ld\,(X)$. Let $ld\,(X) = \tau \geq \aleph_0$ and take an arbitrary point $x = (x_1, x_2, \ldots, x_n) \in X^n$, where $x_i \in X$ for all $i = 1, 2, \ldots, n$.

Then there exist $O_1x_1,\, O_2x_2, \ldots, O_nx_n$ neighborhoods of the $x_1,\, x_2, \ldots, x_n \in X$ points, such that $d\,(O_ix_i) \leq \tau$ for all $i = 1, 2, \ldots, n.$ By the theorem of Hewitt-Marczewski-Pondiceri for the density of topological spaces [5], we have $d\left(\prod\limits_{i=1}^{n} O_ix_i\right) \leq \tau$. The set $\prod\limits_{i=1}^{n} O_ix_i$ is a neighborhood of the point $x = (x_1,\, x_2, \ldots, x_n) \in X^n$, therefore $ld\,(X^n) \leq \tau$. For each point $x \in X$ we have $ld\,(X) \leq \tau$. So $ld\,(X^n) \leq ld\,(X)$ for $n \in N$. Now we shall show reverse inequality $ld\,(X) \leq ld\,(X^n)$. It is clear that the projection $pr : X^n \rightarrow X$ for $pr\,(x) = x_1$, where $x = (x_1,\, x_2, \ldots, x_n) \in X^n$. By the Theorem 2.2 we have that $ld\,(X) \leq ld\,(X^n)$ and so $ld\,(X) = ld\,(X^n)$. Then the equality $ld\,(X) = ld\,(X^n) = ld\,(SP_G^nX)$ is true. Theorem 3.4 is proved. □

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