THE WEAK DENSITY AND THE LOCAL WEAK DENSITY OF SPACE OF THE PERMUTATION DEGREE AND HYPERSPACES

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THE WEAK DENSITY AND THE LOCAL WEAK DENSITY OF SPACE OF THE PERMUTATION DEGREE AND HYPERSPACES

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ABSTRACT. In the work some are studied the weak density and the local weak density of space of the permutation degree and hyperspaces. Proved that for an infinite T_1-space X the followings: a) $wd(X) = wd(SP^n X)$; b) if $Y \subset X$ such that $lwd(Y) = lwd(X)$, then $lwd(SP^n Y) = lwd(SP^n X)$. It also shown that the following: let X be an infinite T_1-space, n positive number, G_1 and G_2 subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then $wd(X) = wd(X^n) = wd(SP^n_{G_1} X) = wd(SP^n_{G_2} X) = wd(SP^n X) = wd(exp_n X)$.

1. INTRODUCTION

In 1981 on the Prague topological symposium V. V. Fedorchuk [1] put forward the following common problems in the theory of covariant functors: Let P be some geometrical or topological property and F- some covariant functor. If X has a property P, then $F(X)$ has the same property P? Or on the contrary, i.e. for what functors, if $F(X)$ possesses a property P, it follows that X possesses the same property P? In our work the property P is the weak density or the local weak density of topological spaces and functors $F = SP^n$, exp: the functor of a permutation degree and the exponential functor, respectively. In 2015 introduced the local weak density of topological spaces [3, 4]. In the work [3] proved that for stratifiable spaces the local density and the local weak density coincide, these

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cardinal numbers are preserved under open mappings, are inverse invariant of a class of closed irreducible mappings. In our work proved that for an infinite T_1-space X the following: let X be an infinite T_1-space and Y is locally weakly τ-dense in X. Then SP^nY is also locally weakly τ-dense in SP^nX.

2. PRELIMINARIES

A permutation group X is the group of all permutations (i.s.one-one and onto mappings $X \to X$. A permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, 3, \dots, n\}$, $S(X)$ is denoted by S_n , as well.

Let X^n be the n-th power of a compact X. The permutation group S_n of all permutations, acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient topology we denote by SP^nX . Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus two points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ are considered to be equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$.The space SP^nX is called the n -permutation degree of a space X. Equivalent relation by which we obtained space SP^nX is called the symmetric equivalence relation. The n-th permutation degree is always a quotient of X^n. Thus, the quotient map is denoted by as following: $\pi_n^s : X^n \to SP^nX$.

Where for every $x = (x_1, \dots, x_n) \in X^n$, $\pi_n^s((x_1, \dots, x_n)) = [(x_1, \dots, x_n)]$ is an orbit of the point $x = (x_1, x_2, \dots, x_n) \in X^n$.

The concept of a permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as group of permutations of coordinates. Consequently, it generates a G -symmetric equivalence relation on X^n. This quotient space of the product of X^n under the G-symmetric equivalence relation is called G-permutation degree of the space X and it is denoted by SP^n_G. An operation $SP^n_G = SP^n$ is also the covariant functor in the category of compacts and it is said to be a functor of G -permutation degree. If $G = S_n$ then $SP^n_G = SP^n$. If the group G consists only of unique element then $SP^n_G = X^n$.

Let X be a T_1-space. The collection of all nonempty closed subsets of X we denote by $expX$. The family B of all sets in the form

$O\langle U_1, \dots, U_n\rangle = \{F : F \in expX, F \subset \bigcup\limits_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, i = 1, 2, \dots, n\}$, where $U_1, \dots, U_n$ is a sequence of open sets of X, generates the topology on the set $expX$. This topology is called the Vietoris topology. The $expX$ with the Vietoris topology is called the exponential space or the hyperspace of X [2]. Let X be a T_1-space. Denote by $exp_n X$ the set of all closed subsets of X cardinality of that is not greater than the cardinal number n, i.e. $exp_n X = \{F \in expX : |F| \leq n\}$.

Let's put $exp_\omega X = \cup\{exp_n X : n = 1, 2, \ldots\}$, $exp_c X = \{F \in expX : F$ is compact in $X\}$.

It is clear, that $exp_n X \subset exp_\omega X \subset exp_c X \subset expX$ for any topological space X. Moreover, if $G_1 \subset G_2$ for subgroups G_1, G_2 of the permutation group $\pi_n^s((x_1, x_2, \ldots, x_n)) = [x = (x_1, x_2, \ldots, x_n)] \in X^n$ then we get a sequence of the factorization of functors:

$X^n \to SP^n_{G_1} X \to SP^n_{G_2} X \to SP^n X \to exp_n X$ [2].

We say that the weak density of the topological space is $\tau \geq \aleph_0$, if τ is the smallest cardinal number such that there exists a π-base coinciding with τ of centered systems of open sets, i.e. there is a π-base $B = \cup\{B_\alpha : \alpha \in A\}$ where B_α is a centered system of open sets for each $\alpha \in A, |A| = \tau$. Weak density of topological space X is denoted by $wd(X)$ [5].

Theorem 2.1. *[5] Let us $\{X_\alpha : \alpha \in A\}$- family of topological spaces such that for each $\alpha \in A$, $wd(X_\alpha) \leq \tau \geq \aleph_0$, where $|A| \leq 2^\tau$. Then we have $wd(\prod\limits_{\alpha \in A} X_\alpha) \leq \tau$.*

Proposition 2.1. *[5] Let X, Y-topological spaces and there exists continuous $f : X \to Y$ "onto" map. Then $wd(Y) \leq wd(X)$.*

Topological space X is said local weak τ-dense at a point x, if τ is the smallest cardinal number such that x has a neighborhood of weak density τ in X. Local weak density at a point x is denoted by $lwd(x)$. The local weak density of a topological space X is defined as the supremum of all numbers $lwd(x)$ for $x \in X$: $lwd(X) = sup\{lwd(x) : x \in X\}$ [3, 4].

Proposition 2.2. *[6] $\pi_n^s : X^n \to SP^n X$ quotient map is an open, closed continuous onto mapping.*

Proposition 2.3. *[7] If X is a topological space, then $exp_n X$ is dense in $expX$.*

Theorem 2.2. *[8] Let X be an infinite T_1-space. Then*

$$wd(X) = wd(exp_n X) = wd(expX).$$

3. Main Results

Theorem 3.1. *Let X be an infinite T_1-space. Then $wd(X) = wd(SP^nX)$.*

Proof. First, we will show that $wd(SP^nX) \leq wd(X)$. Suppose that $wd(X) = \tau \geq \aleph_0$, then by the theorem 2.1 we have $wd(X^n) = \tau$. SP^nX space is a continuous image of the space X^n and so by the Proposition 2.1 this implies that $wd(SP^nX) \leq \tau$.

Now we shall prove that $wd(SP^nX) \geq wd(X)$. Let us $wd(SP^nX) = \tau \geq \aleph_0$. This mains that there exists $SP^nB = \bigcup\{SP^nB_\alpha : \alpha \in A, |A| = \tau\}$ - π-base in SP^nX, where $SP^nB_\alpha = \{SP^nU_s^\alpha : s \in A_\alpha\}$ is centered system of nonempty open sets for each $\alpha \in A$. Let us $B_\alpha = \{(\pi_n^s)^{-1}(SP^nU_s^\alpha) : s \in A_\alpha\}$ and $B = \bigcup\{B_\alpha : \alpha \in A\}$. First we will show that B_α is to be centered system of nonempty open sets in X^n for each $\alpha \in A$. For every finite subfamily $\{SP^nU_{s_i}^\alpha\}_{i=1}^k$ of SP^nB_α we have $\bigcap\limits_{i=1}^{k}\{SP^nU_{s_i}^\alpha\} \neq \emptyset$. Then $\emptyset \neq (\pi_n^s)^{-1}(\bigcap\limits_{i=1}^{k} SP^nU_{s_i}^\alpha) = \bigcap\limits_{i=1}^{k} ((\pi_n^s)^{-1}(SP^nU_{s_i}^\alpha))$. This shows that $B_\alpha = \{(\pi_n^s)^{-1}(SP^nU_s^\alpha) : s \in A_\alpha\}$ is also centered system of nonempty open sets in X^n. Now we will show that B is to be π-base in X^n. Since $SP^nB_\alpha = \{SP^nU_s^\alpha : s \in A_\alpha\}$ is a π-base of SP^nX, for every SP^nU open subset of SP^nX there exists $SP^nU_s^\alpha \in SP^nB_\alpha \subset SP^nB$ such that $SP^nU_s^\alpha \subset SP^nU$. Since the quotient map $\pi_n^s : X^n \to SP^nX$ is open and onto, we have $(\pi_n^s)^{-1}(SP^nU_s^\alpha) \subset (\pi_n^s)^{-1}(SP^nU)$. This means that B is a π-base in X^n and so we have $wd(X^n) \leq \tau$. Theorem 3.1 is proved. □

Corollary 3.1. *If X is an infinite T_1-space and $Y \subset X$ such that $wd(Y) = wd(X)$, then $wd(SP^nY) = wd(SP^nX)$.*

Theorem 3.2. *If X topological space is locally weakly τ-dense, then the product X^n is also locally weakly τ-dense.*

Proof. Take an arbitrary point $x = (x_1, x_2, \dots, x_n) \in X^n$. Since X is locally weakly τ-dense, the $x_i \in X$ has a neighborhood U_i of weakly density $\leq \tau$, for every $i = 1, 2, \dots, n$. The set $\prod\limits_{i=1}^{n} U_i$ is a neighborhood of the point $x \in X^n$. Since $wd(X) \leq \tau$ by the theorem 2.1 we have $wd(\prod\limits_{i=1}^{n} U_i) \leq \tau$. This shows that we have found a weakly τ-dense neighborhood of the point $x \in X^n$. The point x was chosen arbitrary, therefore the product X^n is locally weakly τ-dense. Theorem 3.2 is proved. □

Theorem 3.3. *Let X be an infinite T_1-space and Y is locally weakly τ-dense in X. Then SP^nY is also locally weakly τ-dense in SP^nX.*

Proof. We shall prove this theorem by separating two parts. First, we shall prove that if Y is a subset of X topological space such that, locally weakly τ-dense, then Y^n is also locally weakly τ-dense in the product X^n. That implies from the theorem 3.2 easily.

Now, we shall prove that if Y^n is locally weakly τ-dense in X^n, then SP^nY is also locally weakly τ-dense in SP^nX. Indeed, suppose that X is an infinite T_1-space and $Y^n \subset X^n$ is locally weakly τ-dense. Then for every point $y \in Y^n$ there exists neighbourhood Oy such that Oy is weakly τ-dense in X^n. By the theorem 3.1 $SP^n(Oy) = \{\pi_n^s(y') : y' \in Oy\}$ is also weakly τ-dense in SP^nX. This means that for every point $\pi_n^s(y) \in SP^nY$ there exists $SP^n(Oy)$ such that it is weakly τ-dense in SP^nX. This shows that SP^nY is locally weakly τ-dense in SP^nX. Theorem 3.3 is proved. □

Corollary 3.2. *If X is an infinite T_1-space and $Y \subset X$ such that $lwd(Y) = lwd(X)$, then $lwd(SP^nY) = lwd(SP^nX)$.*

Proposition 3.1. *Let X be an infinite T_1-space, n positive number, G_1 and G_2 subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then $wd(X) = wd(X^n) = wd(SP^n_{G_1}X) = wd(SP^n_{G_2}X) = wd(SP^nX) = wd(exp_nX)$.*

Proof. Let X is an infinite T_1-space. By $X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to exp_nX$ and continuous mappings do not increase the weak density of topological spaces, it directly follows the inequalities

$$wd(X) \geq wd(X^n) \geq wd(SP^n_{G_1}X) \geq wd(SP^n_{G_2}X) \geq wd(SP^nX) \geq wd(exp_nX)$$

and by Theorem 2.2 $wd(X) = wd(exp_nX)$. Hence, we obtain $wd(X) = wd(X^n) = = wd(SP^n_{G_1}X) = wd(SP^n_{G_2}X) = wd(SP^nX) = wd(exp_nX)$. Proposition 3.1 is proved. □

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REFERENCES

[1] V. V. FEDORCHUK: *Covariant functors in the category of compacts, absolute retracts, and $Q-$ manifolds*, Uspekhi Matem. Nauk., **36**(3) (1981), 177–195.

[2] V. V. FEDORCHUK, V. V. FILIPPOV: *Topology of hyperspaces and its applications*, Mathematica, Cybernetica, **4** (1989), 48p.

[3] R. B. BESHIMOV, N. K. MAMADALIEV, F. G. MUKHAMADIEV: *Some properties of topological spaces related to the local density and the local weak density*, Mathematics and Statistics, **3**(4) (2015), 101–105.

[4] R. B. BESHIMOV, F. G. MUKHAMADIEV, N. K. MAMADALIEV: *The local density and the local weak density of hyperspaces*, International Journal of Geometry, **4**(1) (2015), 42–49.

[5] R. B. BESHIMOV: *O note weakly separable spaces*, Mathematica Moravica, **6** (2002), 9–19.

[6] H. WAGNER: *Symmetric, cyclic, and permutation products of manifolds*, Warszawa, 1980.

[7] E. MICHAEL: *Topologies on spaces of supsets*, Trans. Amer. Math. Soc., **1**(71) (1951), 152–172.

[8] R.B. BESHIMOV: *Some cardinal properties of topological spaces connected with weakly density*, Methods of Functional Analysis and Topology, **3**(10) (2004), 17–22.

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