Integration of equation of Kaup system kind with a self-consistent source in the class of periodic functions

Yakhshimuratov A.B.[1], Babadzhanov B.A.[2], Jumaboev R.Sh.[3].
Urgench Branch of Tashkent University of Information Technologies named after Muhammad al-Khwarizmi, Urgench city, Uzbekistan;[1]
Urgench State University, Urgench city, Uzbekistan;[2,3]
e-mail: albaron@mail.ru; a.murod@mail.ru; rjumaboyev95@mail.ru

In [1], D.J. Kaup proved that the nonlinear system of equations

$$\begin{cases} \eta_\tau = \Phi_{xx} + \beta^2 \Phi_{xxxx} - \varepsilon \cdot (\Phi_x \eta)_x \\ \eta = \Phi_\tau + \frac{1}{2}\varepsilon \cdot \Phi_x^2, \end{cases}$$

is completely integrable. The system describes the waves propagation in shallow water. In [2], the complex finite-gap multiphase solutions expressed in terms of the Riemann theta-functions are considered, the multi-soliton solutions are found and the asymptotic behavior of these solutions is studied. In [3, 4] and [5, pp.169-179], the real finite-gap regular solutions of Kaup system were studied. In [6], the 'Inverse Scattering Transform' is used to solve a class of nonlinear equations associated with the inverse problem for the one-dimensional Schrodinger equation with the energy-dependent potential.

It is not difficult (see [2]) to verify that after transformations

$$\eta = \frac{4\beta^2}{\varepsilon}(q + p^2) + \frac{1}{\varepsilon}, \Phi_\tau = \frac{4\beta^2}{\varepsilon}(q + 3p^2) + \frac{1}{\varepsilon}, \Phi_x = -\frac{4\beta}{i\varepsilon}p, t = i\beta\tau,$$

the system of Kaup equations takes a simpler form

$$\begin{cases} p_t = -6pp_x - q_x \\ q_t = p_{xxx} - 4qp_x - 2pq_x. \end{cases}$$

This system we will also call the Kaup system.

The Kaup system can be considered as a compatibility condition (see [2])

$$y_{xxt} - y_{txx} \equiv [(q_t - p_{xxx} + 4qp_x + 2pq_x) + 2\lambda(p_t + 6pp_x + q_x)]y = 0$$

for the system of the linear equations

$$\begin{cases} -y_{xx} + qy + 2\lambda py - \lambda^2 y = 0 \\ y_t + 2\lambda y_x + 2py_x - p_x y = 0. \end{cases}$$

The first of these equations is called the quadratic pencil of Sturm-Liouville equations.

In [7], the Kaup system with self-consistent sources is studied by means of the inverse problem for the quadratic pencil of Sturm-Liouville equations with periodical potential. The inverse problem for the quadratic pencil of Sturm-Liouville equations with periodical potential on the half line and whole line was solved in the works [8-13].

Nonlinear evolution equations with self-consistent sources have received much attention in the recent research literature. Physically, the sources appear in solitary waves with non-constant velocity and lead to a variety of dynamics of physical models. They have important applications in plasma physics, hydrodynamics, solid-state physics, etc. [14-19]. For example, the KdV equation, which is included an integral type self-consistent source, was considered in [17]. By this type equation the interaction of long and short capillary-gravity waves can be described [18]. Other important soliton equations with self-consistent source are the nonlinear Schrodinger equation which describes the nonlinear interaction of an ion acoustic wave in the two component homogeneous plasma with the electrostatic high frequen-cy wave [19]. Other aspects on integration of nonlinear systems are presented in [20-25].

In this paper, the method of the inverse spectral problem for the quadratic pencil of Sturm-Liouville equations with periodic coefficients is used to integrate the equation of Kaup system kind with a self-consistent source in the class of periodic functions. In the one-gap case, we write the explicit formulas for solutions of the problem under consideration, expressed in terms of the Jacobi elliptic functions.

We consider the system of equations with a self-consistent source

$$\begin{cases} p_t = p_{xxx} - 6p_x q - 6pq_x - 30p^2 p_x + \sum_{k=-\infty}^{\infty} \alpha_k(t) s(\pi, \lambda_k, t) (\psi_+^2(x, \lambda_k, t))_x, \\ q_t = q_{xxx} + 6pp_{xxx} + 18p_x p_{xx} - 6qq_x - 24pqp_x - 6p^2 q_x \\ \quad +2\sum_{k=-\infty}^{\infty} \alpha_k(t) s(\pi, \lambda_k, t) \left\{ -p_x \cdot \psi_+^2(x, \lambda_k, t) + (\lambda_k - 2p)(\psi_+^2(x, \lambda_k, t))_x \right\} \end{cases} \tag{1}$$

in the class of real-valued π-periodic on the spatial variable x functions$p = p(x,t)$ and $q = q(x,t)$ which satisfy the regularity of assumptions

$$p(x,t),\ q(x,t) \in C_x^3(t \geq 0) \cap C_t^1(t > 0) \cap C(t \geq 0)$$

with the initial conditions

$$p(x,t)|_{t=0} = p_0(x)\,, \qquad q(x,t)|_{t=0} = q_0(x), \tag{2}$$

Here $p_0(x)$ and $q_0(x)$ are the given real-valued π-periodic functions such that for any nontrivial function $y(x) \in W_2^2[0, \pi]$ satisfying the equalities $y'(0)\bar{y}(0) - y'(\pi)\bar{y}(\pi) = 0$ and $(y, y) = 1$, the following inequality holds:

$$(p_0 y, y)^2 + (q_0 y, y) + (y', y') > 0$$

where $(\cdot\ ,\ \cdot)$ is a scalar product of the space $L_2(0,\ \pi)$. Note that, for $p(x) = 0$ the equation (1) reduces to the Korteweg de Vriez equation with self-consistent source. In the system (1), $\alpha_k(t),\ k \in Z$ is a given sequences of real-valued continuous functions having a uniform asymptotic behavior $\alpha_k = O\left(\frac{1}{k^3}\right),\quad k \to \pm\infty$ and $\psi_+(x, \lambda_k, t)$ is the Floquet solution(normalized by the condition $\psi_+(0, \lambda_k, t) = 1$) of the quadratic pencil of Sturm-Liouville equations

$$T(\lambda, t)y \equiv -y'' + qy + 2\lambda\, py - \lambda^2 y = 0\,, \quad x \in R. \tag{3}$$

Here λ_k is zeros of the function $\Delta^2(\lambda) - 4$, where $\Delta(\lambda) = c(\pi, \lambda,\ t) + s'(\pi, \lambda,\ t)$. We denote by $c(x, \lambda, t)$ and $s(x, \lambda, t)$ the solutions of equation (3) satisfying the initial conditions $c(0, \lambda, t) = 1$, $c'(0, \lambda, t) = 0$ and $s(0, \lambda, t) = 0$, $s'(0, \lambda, t) = 1$, respectively.

The spectrum of the quadratic pencil (3) is real and coincides with the set [8-9]

$$\sigma(T)=\{\lambda\in R|-2\le\Delta(\lambda)\le 2\}=R\backslash\bigcup_{n=-\infty}^{\infty}(\lambda_{2n-1},\ \lambda_{2n}).$$

The intervals $(\lambda_{2n-1},\ \lambda_{2n})$, $n\in Z$ are called the gaps or lacunas. The numbering is introduced such that $\lambda_{-1}<0<\ \lambda_0$.

We denote by $\xi_n(t)$, $n\in Z\backslash\{0\}$ the eigenvalues of the Dirichlet problem ($y(0)=y(\pi)=0$) for equation (3). The inclusions $\xi_n(t)\in[\lambda_{2n-1},\ \lambda_{2n}]$ and the equality

$$s(\pi,\ \lambda,\ t)=\pi\prod_{0\ne k=-\infty}^{k=\infty}\frac{\xi_k(t)-\lambda}{k} \tag{4}$$

are fulfilled.

The numbers $\xi_n=\xi_n(t)$ with the signs $\sigma_n=\sigma_n(t)=sign\,\{s'(\pi,\xi_n)-c(\pi,\xi_n)\}$, $n\in Z\backslash\{0\}$ are called the spectral parameters of the quadratic pencil (3).

The boundaries λ_n of the spectrum and the spectral parameters ξ_n, σ_n are called the spectral data of problem (3).

The aim of this work is to develop a procedure for constructing the solution ($p(x,t)$, $q(x,t)$, $\psi_+(x,\lambda_k,t)$) of problem (1)-(3) by means of the inverse spectral problem for the quadratic pencil of Sturm-Liouville equations (3).

The main result of the paper is included in the theorem below.

Theorem. Let $p(x,t)$, $q(x,t)$ and $\psi_+(x,\lambda_k,t)$ be solution of problem (1)-(3). Then the spectrum of problem (3) does not depend on t, and the spectral parameters $\xi_n(t)$, $n\in Z\backslash\{0\}$ satisfy the analogue of the system of Dubrovin equations

$$\begin{aligned}&\dot{\xi}_n(t)=2(-1)^n\sigma_n(t)sign(n)\cdot\sqrt{(\xi_n(t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t))}\times\\&\times\sqrt{(\xi_n(t)-\lambda_{-1})(\xi_n(t)-\lambda_0)\prod_{k\ne n,0}\frac{(\xi_n(t)-\lambda_{2k-1})(\xi_n(t)-\lambda_{2k})}{(\xi_n(t)-\xi_k(t))^2}}\times\\&\times\left[\left\{4\xi_n(t)+2(\lambda_{-1}+\lambda_0)+4\sum_{0\ne k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2}-\xi_k(t)\right)\right\}\xi_n(t)+\right.\\&+2\left\{\frac{\lambda_{-1}+\lambda_0}{2}+\sum_{0\ne k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2}-\xi_k(t)\right)\right\}^2+\\&\left.+(\lambda_{-1})^2+(\lambda_0)^2+\sum_{0\ne k=-\infty}^{\infty}\left((\lambda_{2k-1})^2+(\lambda_{2k})^2-2\xi_k^2(t)\right)+\sum_{k=-\infty}^{\infty}\frac{\alpha_k(t)s(\pi,\lambda_k,t,\tau)}{\xi_n(t)-\lambda_k}\right].\end{aligned} \tag{5}$$

The sign $\sigma_n(t)=\pm1$ changes at each collision of the point $\xi_n(t)$ with the boundaries of its gap $[\lambda_{2n-1},\ \lambda_{2n}]$. Moreover, the following initial conditions are fulfilled:

$$\xi_n(t)|_{t=0}=\xi_n^0,\quad \sigma_n(t)|_{t=0}=\sigma_n^0,\ n\in Z\backslash\{0\}, \tag{6}$$

where ξ_n^0, σ_n^0, $n \in Z\backslash\{0\}$ are the spectral parameters of the quadratic pencil of Sturm-Liouville equations corresponding to the coefficients $p_0(x)$ and $q_0(x)$.

Corollary 1. If instead of $p(x,t)$ and $q(x,t)$ we consider the functions $p(x+\tau,t)$ and $q(x+\tau,t)$, then, as seen from the previous section, the eigenvalues of the periodic and antiperiodic problems do not depend on the parameters τ and t. However, the eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ and t: $\xi_n = \xi_n(\tau,t)$, $\sigma_n = \sigma_n(\tau,t) = \pm 1$.

Corollary. The theorem gives a method for solving problem (1)-(3). First we find the spectral data λ_n, $n \in Z$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \in Z\backslash\{0\}$ of the quadratic pencil of Sturm-Liouville equations corresponding to the coefficients $p_0(x+\tau)$ and $q_0(x+\tau)$. Then we solve the Cauchy problem $\xi_n(\tau,t)\,|_{t=0} = \xi_n^0(\tau)$, $\sigma_n(\tau,t)\,|_{t=0} = \sigma_n^0(\tau)$,$n \in Z\backslash\{0\}$ for Dubrovin system (5). Finally, by using the trace formulas

$$p(\tau,t) = \frac{\lambda_{-1}+\lambda_0}{2} + \sum_{0\neq k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2} - \xi_k(\tau,t)\right),$$

$$q(\tau,t) + 2p^2(\tau,t) = \frac{(\lambda_{-1})^2+(\lambda_0)^2}{2} + \sum_{0\neq k=-\infty}^{\infty}\left(\frac{(\lambda_{2k-1})^2+(\lambda_{2k})^2}{2} - \xi_k^2(\tau,t)\right)$$

we get the expressions of $p(\tau,t)$ and $q(\tau,t)$. After that the Floquet solutions $\psi_+(x,\lambda_k,t)$ of equation (3) can be found easily.

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