ON GEOMETRY OF VECTOR FIELDS

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Abstract. It is well known that the study of the geometry and topology of the attainability set of a family of vector fields is one of the main tasks of the qualitative control theory, which is closely related to the geometry of orbits of vector fields. In this paper, we present the authors' results on the geometry of the attainability set of a family of vector fields: results on the geometry of T-attainability sets and the geometry of orbits of Killing vector fields.

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1. Introduction. In this paper, we examine the geometry of orbits of families of smooth vector fields defined on a smooth manifold and the geometry of the attainability set. We review the authors' results obtained in [12, 13].

The structure of orbits of families of smooth vector fields was studied by many mathematicians due to its importance in applications, in the theory of optimal control of dynamic systems, in geometry, and in the theory of foliations (see [1, 2, 6, 12, 16, 17]).

In the qualitative control theory, the controllability set (or the attainability set) of a control system on a smooth manifold in the class of piecewise-constant controls coincides with the negative (positive) orbit of the family of vector fields, which is determined uniquely by the control system. In the case of symmetric systems, the controllability set (and also the attainability set) coincides with the orbit.

On the other hand, any family of vector fields determines a dynamic polysystem. Thus, the study of the structure of the controllability set is closely related to the study of the structure of orbits of a family of vector fields. It is well known that the controlability set is one of the main objects of the qualitative optimal control theory.

In the study of qualitative properties of control systems on smooth manifolds (for example, controllability issues), the opportunity to apply differential-geometric methods arises (these methods are discussed in [1]).

2. Preliminaries. Let M be a smooth (of class C^∞) manifold of dimension n and $V(M)$ be the set of all smooth (of class C^∞) vector fields on M. We denote by $[X, Y]$ the Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with respect to the Lie bracket.

For a set $D \subset V(M)$, we denote by $V(D)$ the smallest Lie subalgebra containing the set D, and by $L(x)$ the orbit of the family D containing the point $x \in M$.

We denote by $P(x)$ the linear span of the set of vectors $D(x) = \{X(x) : X \in D\}$ and introduce the following subspace of the tangent space $T_x M$ at the point x:

$$V_x(D) = \{X(x) : X \in V(D)\}$$

Consider the mappings $P : x \to P(x)$ and $P_D : x \to V_x(D)$ that assign some subspaces $P(x)$ and $V_x(D)$ of the tangent space $T_x M$, respectively, to each point x. Such mappings are called *distributions*.

R. Hermann was the first to point out the importance in the control theory of the following Zhou result presented in [3] (this result was almost simultaneously proved by P. Rashevsky in [15]).

Assertion 1. If the dimensions $\dim P(x)$ and $\dim V_x(D)$ are independent of x, then for each point x the set $L(x)$ is an integral submanifold of the distribution P_D.

In the case where the dimension of the linear space $V_x(D)$ is not constant, Hermann obtained sufficient conditions under which the orbit $L(x)$ is an integral submanifold of a completely integrable distribution $x \to V_x(D)$ for each point $x \in M$.

The fundamental result in this direction is the following theorem.

Theorem 1 (Sussmann [17]). *If a manifold M and vector fields from D belong to the class C^∞, then for each $x \in M$, the orbit $L(x)$ is an immersed submanifold of M.*

In other words, there exists a completely integrable distribution on M and, moreover, for each point $x \in M$, the orbit $L(x)$ coincides with the maximal integral submanifold of this distribution passing through the point x.

P. Stefan proved this result in the case where M and the vector fields from D have smoothness of the class C^r, $r \geq 1$ (see [16]). It is known that in the case where the vector fields of D belong to the class C^0, then the orbit is not a manifold: a continuous vector field in the case where the solution of the corresponding differential equation is not unique can serve as an example.

The results of [9] imply that if M and vector fields from D are analytic, then the distribution $P_D : x \to V_x(D)$ is completely integrable and, moreover, each orbit is an integral submanifold of P_D. Thus, in this case, the dimension of the orbit $L(x)$ is equal to $\dim V_x(D)$ for all $x \in M$. In the general case, the following relation holds for all $x \in M$:

$$\dim V_x(D) \leq \dim L(x).$$

3. Geometry of the attainability set of vector fields. Consider the set $D \subset V(M)$, which can contain finite or infinite number of smooth vector fields.

For a point $x \in M$, we denote by $t \to X^t(x)$ the integral curve of the vector field X passing through the point x for $t = 0$. The mapping $t \to X^t(x)$ is defined in some domain $I(x) \subset \mathbb{R}$, which in general depends on the field X and the initial point x.

In the sequel, in formulas of the form $X^t(x)$ we assume that $t \in I(x)$. If for all points $x \in M$ the domainf $I(x)$ of the curve $t \to X^t(x)$ coincides with the real axis, then the vector field X is said to be *complete*. In this case, the flow of this vector field generates a dynamical system.

Definition 1. The orbit $L(x)$ of a family D of vector fields passing through a point x is defined as the set of points $y \in M$ for which there exist real numbers $t_1, t_2, \ldots, t_k$, and vector fields $X_1 X_2, \ldots, X_k$ from D, $k \in \mathbb{N}$, such that

$$y = X_k^{t_k}\left(X_{k-1}^{t_{k-1}}\left(\ldots\left(X_1^{t_1}(x)\right)\ldots\right)\right). \tag{1}$$

Definition 2. The point $y \in L(x)$ from Eq. (1) is said to be *T-attainable* from a point $x \in M$ if $\sum_i t_i = T$.

We denote by $A_x(T)$ the set of all points that are T-attainable from the point x.

Recall that a submanifold $N \subset M$ is said to be *immersed* in M if the canonical injection $i : N \to M$ is a differentiable mapping of maximal rank.

The topology of the orbit $L(x)$ (the Sussman topology) is introduced as the strongest topology for which all mappings of the form

$$\left(t_1, t_2, \ldots, t_k\right) \in \mathbb{R}^k \to X_k^{t_k}\left(X_{k-1}^{t_{k-1}}\left(\ldots\left(X_1^{t_1}(x)\right)\ldots\right)\right)$$

are continuous; here $t_1, t_2, \ldots, t_k$ are real numbers and $X_1, X_2, \ldots, X_k$ are vector fields of the family D.

The proper topology of the orbit as an immersed submanifold is stronger than the topology induced from M. For example, for the irrational winding of the torus, these topologies are different for all trajectories.

It was proved in [17] that an orbit is a smooth manifold. Based on the same idea, we proved in [13] the following theorem on the geometry of the set of T-attainable points.

Theorem 2. *For each $x \in M$ and any T, the set $A_x(T)$ is an immersed submanifold of the orbit $L(x)$ of codimension 1 or 0.*

Another important contribution of Sussmann in the geometry of the attainability set is the following theorem.

Theorem 3 (see [8]). *Let M be a smooth connected manifold of dimension n. There exists a system D consisting of two vector fields such that $L^+(x) = M$ for each point $x \in M$.*

Using Theorem 3, we proved in [13] the following assertion.

Theorem 4. *Let M be a smooth connected manifold of dimension $n \geq 2$. There exists a system D consisting of three vector fields such that $A_x(0) = M$ for each point $x \in M$.*

For manifolds with nonzero Euler characteristic, the following result is valid.

Theorem 5 (see [13]). *Let M be a smooth, compact, connected manifold of dimension $n \geq 2$ whose Euler characteristic is nonzero. There exists a system D consisting of two vector fields such that $A_x(0) = M$ for each point $x \in M$.*

The following example shows that on a compact connected manifold M with zero Euler characteristic, a system D consisting of two vector fields can exist such that $A_x(0) = M$ for each point $x \in M$.

Consider the three-dimensional sphere $S^3 \subset \mathbb{R}^4$

$$x^2 + y^2 + z^2 + w^2 = 1,$$

where x, y, z, and w are the Cartesian coordinates in $\mathbb{R}^4$, and the following system of two vector fields on S^3:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial z}, \quad Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}.$$

It is easy to verify that these vector fields are Killing fields, i.e., the local diffeomorphisms $x \to X^t(x)$ and $x \to Y^t(x)$ are isometries of the sphere S^3 for each t.

The Lie bracket $[X, Y]$ of the vector fields X and Y has the following form:

$$[X, Y] = -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w}.$$

The vector fields X, Y, and $[X, Y]$ belong to the Lie subalgebra $V(D)$, which is the minimal Lie subalgebra of the Lie algebra $V(M)$ containing the set D.

At the point $p(1, 0, 0, 0) \in S^3$, the vectors $X(p)$, $Y(p)$, and $[X, Y](p)$ are linearly independent, i.e., the subspace $V_p(D) = \{X(p) : X \in V(D)\}$ is three-dimensional. Therefore, the orbit $L(p)$ is also three-dimensional. Due to the fact that X and Y are Killing vector fields, the orbit $L(p)$ is a closed subset of $\mathbb{R}^4$ (and hence in S^3; see [12]). On the other hand, as follows from the proof of Theorem 1, due to the maximality of dimension, the orbit $L(p)$ is an open subset of S^3. Hence, the orbit coincides with S^3.

Now let us consider the sets $A_q(0)$, $q \in S^3$. If the sets A_q are submanifolds of codimension 1, due to the fact that the vector fields X and Y are Killing vector fields, they generate a two-dimensional

Riemannian foliation on S^3 (see [12]). As follows from [11], there are no two-dimensional Riemannian foliations on the three-dimentional sphere. Hence, the set $A_q(0)$ coincides with S^3 for all $q \in S^3$.

For symmetric systems, the following theorem holds.

Theorem 6 (see [13]). *Let a system D be symmetric and contain a complete vector field. Then for each $T \in \mathbb{R}$ and each points $x \in M$, the following equality holds*:

$$A_x(T) = L(x).$$

Recall that a system D of vector fields is said to be *symmetric* if $X \in D$ implies $-X \in D$.

In the following example, the sets $A_q(0)$ are submanifolds of the orbit $L(p)$ of codimension 1.

Let $M = \mathbb{R}^3$ and let D consist of the vector fields

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial z}.$$

In this case, for each point $p(x, y, z) \in M$ such that $x^2 + y^2 > 0$, the orbit $L(p)$ is a cylinder and for each point $q \in L(p)$, the set $A_q(0)$ is a cylindrical helix whose tangent field is

$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

For points of the axis OZ, the orbit $L(p)$ and the set $A_q(0)$ coincides with the axis OZ.

4. Geometry of Killing vector fields.

Now we turn to the study of the geometry of Killing vector fields. In this section, we present results obtained in [12].

Recall that a vector field X on M is called a *Killing vector field* if the one-parameter group of local transformations $x \to X^t(x)$ generated by the field X consists of isometries.

Recall also that a mapping P that assigns a subspace $P(x) \subset T_x M$ to each point $x \in M$ is called a *distribution*. If $\dim P(x) = k$ for all $x \in M$, then P is called a k-dimensional distribution.

A family D of smooth vector fields naturally generates a smooth distribution, which to each point $x \in M$, assigns a subspace $P(x)$ of the tangent space $T_x M$ generated by the following set of vectors:

$$D(x) = \{X(x) : X \in D\}.$$

Of course, the dimensions of the subspaces $P(x)$ can vary from point to point.

A connected submanifold N of a manifold M is called an *integral submanifold* of the distribution P (or P_D) if for any point $x \in N$, the following equality is valid:

$$T_x N = P(x) \quad \left(T_x N = A_x(D)\right).$$

A distribution P is said to be *completely integrable* if for each point $x \in M$, there exists a submanifold N_x of the manifold M such that $T_y N_x = P(y)$ for all $y \in N_x$. A submanifold N_x is called an *integral submanifold* of the distribution P.

For a vector field X, we write $X \in P$ if $X(x) \in P(x)$ for all $x \in M$. A distribution P is said to be *involutive* if the fact that $X, Y \in P$ implies that $[X, Y] \in P$, where $[X, Y]$ is the Lie bracket of the vector fields X and Y.

A necessary and sufficient condition of the complete integrability of a distribution of constant dimension is as follows.

Theorem 7 (see 7). *A distribution P on a manifold M is completely integrable if and only if it is involutive.*

We show that if a set D consists of Killing vector fields, then the distribution $P_D : x \to V_x(D)$ is also completely integrable in the case where $M = \mathbb{R}^n$, and orbits of the family D are closed subsets.

We note that the Lie bracket of two Killing fields again gives a Killing field and a linear combination of Killing fields over the field of real numbers is also a Killing field. Therefore, the set $K(M)$ of all

Killing vector fields on a manifold M is a Lie algebra over the field of real numbers. In addition, it is well known that the Lie algebra $K(M)$ of Killing vector fields on a connected Riemannian manifold M has dimension not greater than $n(n+1)/2$, $n = \dim M$. If $\dim K(M) = n(n+1)/2$, then M is a manifold of constant curvature (see [7]).

We denote by $A(D)$ the smallest Lie subalgebra of the Lie algebra $K(M)$ containing the set D.

Since the algebra $K(M)$ is finite-dimensional, then there exist vector fields $X_1, X_2, \ldots, X_m$ from $A(D)$ such that the vectors $X_1(x), X_2(x), \ldots, X_m(x)$ form a basis of the subspace $V_x(D)$ for each $x \in M$.

The Frobenius theorem generalized by Hermann for distributions of nonconstant dimension gives a necessary and sufficient condition of the complete integrability of families of vector fields that consisting of a finite number of vector fields.

Theorem 8 (see [4]). *Let $D = \{X_1, X_2, \ldots, X_k\}$ be a family of vector fields on a manifold M. The family D generates a completely integrable distribution if and only if it is involutive.*

The involutivity of a family of vector fields $D = \{X_1, X_2, \ldots, X_k\}$ means the following: for each $X, Y \in D$, there exist smooth functions $f^l(x)$, $x \in M$, $l = \overline{1, k}$, such that

$$[X, Y] = \sum_{l=1}^{k} f^l(x) X_l.$$

Thus, in the case where the family D consists of Killing vector fields, Theorem 8 implies the following assertion.

Theorem 9. *Any orbit of a family D is an integral submanifold of the completely integrable distribution $P_D : x \to V_x(D)$.*

Theorem 10. *Let $M = \mathbb{R}^n$ and let a set D consist of Killing vector fields. Then each orbit of the family D is a closed subset.*

We recall some concepts from the theory of foliations.

Let $f : M \to N$ be a differentiable mapping of maximal rank, where M is a smooth Riemannian manifold of dimension n and N is a smooth Riemannian manifold of dimension m, where $n > m$. Then for each point $q \in N$, the set $L_q = \{p \in M : f(p) = q\}$ is a manifold of dimension $n - m$, and the partition of M into the manifolds L_q is a k-dimensional foliation, where $k = n - m$.

Let L be a leaf of the foliation F (i.e., an orbit of the family D), $x \in L$, $T_x L$ be the tangent space L at a point x, and $H(x)$ be the orthogonal complement of $T_x L$.

Consider the subbundles $TF : x \to T_x L$ and $H : x \to H(x)$ of the tangent bundle TM of the manifold M arise. In this case, each vector field $X \in V(M)$ can be represented as $X = X_F + X_H$, where X_F and X_H are orthogonal projections of X on TF and H, respectively. If $X_H = 0$, then it is called a *vertical field* (tangent to F), and if $X_F = 0$, then X is called a *horizontal field*.

A mapping $f : M \to N$ is called a *Riemannian submersion* if the differential df of the mapping f preserves the lengths of horizontal vectors (see [14]).

We denote by $B = M/F$ the set of fibers F endowed with the factor topology. We consider the mapping $\pi : M \to F$ such that $\pi(x) = L(x)$, where $L(x)$ is a fiber containing the point x. The following theorem shows that orbits are fibers of the Riemannian submersion.

Theorem 11. *Let $M = \mathbb{R}^n$, let a set D consist of Killing vector fields, and $\dim V_x(D) = k$ for all $x \in M$, where $0 < k < n$. Then the set of fibers $B = M/F$ endowed with the factor topology has the differential structure of a smooth $(n-k)$-dimensional manifold such that the mapping $\pi : M \to B$ is a smooth Riemannian submersion.*

Corollary 1. *The manifold B is a manifold of nonnegative curvature.*

Theorem 12. *Under the conditions of Theorem 11, orbits of the family D are parallel planes if and only if the manifold $B = M/F$ is a manifold of zero curvature.*

The following example shows that in the general case, the distribution $H : x \to H(x)$ is not always completely integrable even in the case $M = \mathbb{R}^n$. Let us consider the following vector field in $M = \mathbb{R}^3$:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

This field is a Killing vector field and its integral curves are cylindrical helixes.

The orthogonal distribution $H : p \to H(p)$ is given at each point by the following horizontal vector fields:

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

We consider the vector fields $Y_* = d\pi(Y)$ and $Z_* = d\pi(Z)$ on $B = M/F$. By virtue of the fact that the mapping $\pi : M \to B$ has the maximal rank, the vector fields Y_* and Z_* are linearly independent at each point of the manifold $B = M/F$.

We calculate the sectional curvature of the manifold $B = M/F$ in the two-dimensional direction determined by the vectors $Y_*(q)$ and $Z_*(q)$ at a point $q \in B$. By the O'Neil formula, if (x, y, z) are the Cartesian coordinates of a point $p \in \pi^{-}1(q)$, then for the vertical component $[Y, Z]^v$ of the Lie bracket $[Y, Z]$ of vector fields, the following equality holds:

$$[Y, Z]^v(p) = \lambda X(p),$$

where $\lambda = (x^2 + y^2 + 1)^{-1}$. Hence we obtain the following expression for the curvature:

$$K_*(Y_*, Z_*)(q) = \frac{3}{(x^2 + y^2 + 1)^2}.$$

Thus, in this case the manifold $B = M/F$ is a two-dimensional manifold of strictly positive curvature. In this example, all integral curves (orbits) passing through the origin (except for one) are not one-dimensional planes.

It follows from Theorem 2 that the manifolds $A_y(0)$ for points $y \in L(x)$ either coincide with $L(x)$ or generate a foliation of codimension 1 on $L(x)$. This allows one to apply the methods of foliation theory for the study of geometry of the manifolds $A_y(0)$. It is known that if a system consists of Killing vector fields, then this foliation is a Riemannian foliation (for geometry of orbits of Killing vector fields, see [12].

Recall that a foliation F is said to be *Riemannian* if each geodesic orthogonal at some point to the fiber of the foliation F remains orthogonal to all fibers F at all of its points (see [12]).

Theorem 13. *Let $M = \mathbb{R}^n$, D consist of Killing vector fields, and for a point $p \in M$, let the orbit of $L(p)$ be a k-dimensional plane, $0 \leq k \leq n$. Then for all points $q \in L(p)$, the sets $A_q(0)$ either coincide with $L(p)$ or are parallel hyperplanes in $L(p)$.*

Indeed, if submanifolds $A_q(0)$ do not coincide with the orbit $L(p)$, then they generate a Riemannian foliation of codimension 1 on $L(p)$ (see [12]). As follows from [10], a Riemannian foliation of codimension 1 of Euclidean space consists of parallel planes.

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