

On the geometry of Hamiltonian vector fields

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Abstract. It is known that vector fields play an important role in many areas of mathematics and technology, in particular, in the theory of dynamical systems. Hamiltonian dynamical systems are generated by Hamiltonian vector fields. The paper studies the geometry of a singular foliation of a four-dimensional Euclidean space generated by the orbits of two Hamiltonian vector fields. It is proved that the regular leaf of this foliation is a surface of positive normal curvature and nonzero normal torsion.

1 Introduction

Let M be a Riemannian (smooth) manifold of dimension n .

Definition 1. A partition F of a Riemannian manifold M by path-connected immersed submanifolds L_α is called a singular foliation of M if it verifies condition:

for each leaf L_α and each vector $v \in T_p L_\alpha$ at the point p there is $X \in XF$ such that $X(p) = v$, where $T_p L_\alpha$ is the tangent space of the leaf L_α at the point p , XF is the module of smooth vector fields on M tangent to leaves (XF acts transitively on each leaf) [1-4].

If the dimension of L is maximal, it is called regular, otherwise L is called singular. It is known that orbits of vector fields generate singular foliation.

Let us consider a set of vector fields $D \subset V(M)$ of the Lie algebra of all smooth (class C^∞) vector fields $V(M)$ and the smallest Lie sub algebra containing D by $A(D)$. Let $t \rightarrow X^t(x)$ be an integral curve of the vector field X with the initial point x for $t = 0$, which is defined in some region $I(x)$ of real line.

Definition 2. The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, \dots, t_k \in R$ and vector fields $X_{i_1}, X_{i_2}, \dots, X_{i_k} \in D$ such that

$$y = X_{i_k}^{t_k} (X_{i_{k-1}}^{t_{k-1}} (\dots (X_{i_1}^{t_1}) \dots))$$

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where k is an arbitrary positive integer.

The fundamental result in study of orbits is Sussmann theorem [4], which asserts that every orbit is an immersed submanifold of M .

By setting, $A_x(D) = \{X(x) : X \in A(D)\}$ we obtain an involutive distribution $P_D : x \rightarrow A_x(D)$ If $\dim A_x(D) = \text{const}$ for every x then the distribution $P_D : x \rightarrow A_x(D)$ is completely integrable by the Frobenius theorem. If the dimension $\dim A_x(D)$ depends on x , then we can use Hermann theorem for distributions of variable dimension, which gives condition for the complete integrability of distributions which is finitely generated [5, 6].

Definition 3. A system of vector fields $D = \{X_1, X_2, \dots, X_k\}$ on M is in involution if there exist smooth real-valued functions $f_{ij}^l(x), x \in M, i, j, l = 1, \dots, k$ such that for each (i, j) it takes

$$[X_i, X_j] = \sum_{l=1}^k f_{ij}^l X_l$$

Hermann Theorem [6]. In order a system of smooth vector fields on M $D = \{X_1, X_2, \dots, X_k\}$ to generate completely integrable distribution, it is necessary and sufficient that it be involutive.

Let us recall some notions from geometry of Hamilton systems [3,6]. Given a smooth manifold M a Poisson bracket on M assigns to each pair of smooth, real-valued functions $\{F, H\} : M \rightarrow R$ another smooth, real-valued function, which we denote by $\{F, H\}$.

There are certain basic properties that such a bracket operation must satisfy in order to qualify as a Poisson bracket.

Definition 4. A Poisson bracket on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

1. Bilinearity: $\{\lambda F + \mu P, H\} = \lambda\{F, H\} + \mu\{P, H\}$,
 $\{F, \lambda H + \mu P\} = \lambda\{F, H\} + \mu\{F, P\}$. $\lambda, \mu \in R$
2. Skew-Symmetry: $\{F, H\} = -\{H, F\}$
3. Jacobi Identity: $\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$
4. Leibniz' Rule: $\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$

(Here \cdot denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called a Poisson manifold, the bracket defining a Poisson structure on M . The notion of a Poisson manifold is slightly more general than that of a symplectic manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even-dimensional. This is borne out by the standard examples from classical mechanics.

Let M be the Euclidean space R^m , with coordinates

$$(p, q, z) = (p^1, \dots, p^n, q^1, \dots, q^n, z^1, \dots, z^l),$$

where $m = 2n + l$, if $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^n \left\{ \frac{\partial H}{\partial p^i} \cdot \frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i} \cdot \frac{\partial F}{\partial p^i} \right\}$$

This bracket is clearly bilinear and skew-symmetric; the verifications of the Jacobi identity and the Leibniz rule are simple exercises.

We note the particular bracket identities:

$$\begin{aligned} \{p^i, p^j\} &= 0, \{q^i, q^j\} = 0, \{q^i, p^j\} = \delta_j^i. \\ \{p^i, z^k\} &= \{q^i, z^k\} = \{z^t, z^k\} = 0. \end{aligned}$$

in which i and j run from 1 to n , when t and k run from 1 to l . δ_j^i is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 5. Let M be a Poisson manifold and $H : M \rightarrow R$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgradH$ on M satisfying

$$sgradH(F) = \{F, H\} = -\{H, F\}$$

for every smooth function $F : M \rightarrow R$.

In the case of the canonical Poisson bracket on R^m $m = 2n + l$, the Hamiltonian vector field corresponding to $H(p, q, z)$ is clearly

$$sgradH = \sum_{i=1}^n \left(\frac{\partial H}{\partial p^i} \cdot \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \cdot \frac{\partial}{\partial p^i} \right)$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \frac{dz^l}{dt} = 0$$

There is a fundamental connection between the Poisson bracket of two functions and the Lie bracket of their associated Hamiltonian vector fields, which forms the basis of much of the theory of Hamiltonian systems. It is well known following theorem:

Theorem 1. Let M be a Poisson manifold and $\{F, H\} : M \rightarrow R$ be smooth functions with corresponding Hamiltonian vector fields $sgradF, sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:

$$sgrad\{F, H\} = [sgradF, sgradH]$$

Definition 6. Let M^{2n} be a Poisson manifold and $sgradH$ Hamiltonian vector field with a smooth Hamiltonian function H .

Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville*, if exists set of smooth functions f_1, \dots, f_n as:

- 1) f_1, \dots, f_n are first integrals of $sgradH$ Hamiltonian vector field,
- 2) they are functionally independent on M , that is, almost everywhere on M their gradients are linearly independent,
- 3) $\{f_i, f_j\} = 0$ for any i and j ,

- 4) the vector fields $sgrad f_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line.

Definition 7. Partition of the manifold M^{2n} into connected components of joint level surfaces of the integrals f_1, \dots, f_n is called *The Liouville foliation* corresponding to the completely integrated system [6]

Since f_1, \dots, f_n is preserved by $sgrad H$, each leaf of the Liouville foliation is invariant surface. Liouville foliation is consists of regular leaves (which fill almost all M) and special leaves (a subset of zero measure).

2 On the geometry of Hamiltonian vector fields

Let us consider a family of $D = \{X_1, X_2\}$ vector fields on four-dimensional Euclidean space E^4 with Cartesian coordinates p_1, p_2, q_1, q_2 where

$$X_1 = q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2}, X_2 = q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} \quad (1)$$

Theorem 2. The family of orbits of the vector fields (1) is a singular foliation regular leaf of which is a surface with a positive normal curvature and with nonzero normal torsion.

Proof. First, we note that for Lie bracket of the vector field X_1, X_2 it holds $[X_1, X_2] = 0$. follows from Hermann theorem this system generates completely integrable distribution

The vector field

$$X_1 = q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2},$$

generates following one parametrical group of transformations

$$(p_1, p_2, q_1, q_2) \rightarrow (p_1cht + q_1sht, p_2cht + q_2sht, p_1sht + q_1cht, p_2sht + q_2cht),$$

where $t \in R$

The vector field

$$X_2 = q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}$$

generates following one parametrical group of transformations:

$$(p_1, p_2, q_1, q_2) \rightarrow (p_1chs + q_1shs, p_2, p_1shs + q_1chs, q_2), s \in R$$

We need to find the invariant functions of the groups generated by vector fields. It is easy to check the following equalities

$$X_1(I_1) = 0, X_1(I_2) = 0, X_2(I_1) = 0, X_2(I_2) = 0$$

for the functions:

$$I_1(p_1, p_2, q_1, q_2) = p_1^2 - q_1^2, I_2(p_1, p_2, q_1, q_2) = p_2^2 - q_2^2$$

It follows the functions I_1, I_2 are invariant functions [6].

This invariant functions give us a family of two dimensional surfaces

$$p_1^2 - q_1^2 = C_1, p_2^2 - q_2^2 = C_2, \quad (2)$$

where C_1, C_2 are constants.

Let L is a regular connected surface defined by equations (2). For a point $p \in L$ the orbit $L(p)$ of the family of vector fields contained in L . According to Sussmann theorem [4,7] the orbit $L(p)$ is a two-dimensional manifold and therefore it is an open subset of L . This implies that the surface L is a union of orbits. Due to the connectedness of L , we get $L(p) = L$.

Now we can check metric characteristics of the surface L .

Let us recall some notions on the geometry of two dimensional surface in a four-dimensional Euclidean space E^4 .

Consider on a two-dimensional surface F at the point x some direction given by non-zero vector t . The vector t and the normal plane N of the surface at the point x define a hyper plane $E^3(x, t, N)$ in E^4 that intersects over the validity of F along some curve γ . The curve γ is called the normal section at the point x in the direction t . According to its construction the curve γ is a three-dimensional curve. Curvature $k_N(x, t)$ and torsion $\chi_N(x, t)$ of the curve γ at the point x is called, respectively, the normal curvature and normal torsion of the surface at the point x in the direction t . Geometry of two dimensional surfaces in the four dimensional Euclidean space is studied by many authors [8-10].

We will parameterize the surface by the following equations

$$\begin{cases} p_1 = \sqrt{|c_1|} chu \\ p_2 = \sqrt{c_2} chv \\ q_1 = \sqrt{|c_1|} shu \\ q_2 = \sqrt{c_2} shv \end{cases}$$

Now we find

$$\begin{aligned} \frac{\partial r}{\partial u} = r_1 &= \{\sqrt{|c_1|} shu; 0; \sqrt{|c_1|} chu; 0\}, \\ \frac{\partial r}{\partial v} = r_2 &= \{0; \sqrt{c_2} shv; 0; \sqrt{c_2} chv\} \end{aligned}$$

and coefficients of first quadratic form

$$g_{11} = |c_1| ch2u, \quad g_{12} = g_{21} = 0, \quad g_{22} = \langle r_2, r_2 \rangle = |c_2| ch2v$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

We need two normal vectors to find coefficients of second quadratic forms

$$n_1 = \left\{ -\frac{chu}{ch2u}; 0; -\frac{shu}{ch2u}; 0 \right\}$$

and

$$n_2 = \left\{ 0; -\frac{chv}{ch2v}; 0; \frac{shv}{ch2v} \right\}.$$

Now we can find two-second quadratic forms of the regular surface. Coefficients of the first of them is calculated by the formula

$$b_{ij} = -\frac{1}{|n_1|} \langle \partial_i r, \partial_j n_1 \rangle.$$

By using these equation we find that

$$b_{11} = -\frac{\sqrt{|c_1|}}{ch2u}, b_{12} = b_{21} = b_{22} = 0$$

Coefficients of the second of them are calculated by the formula

$$c_{ij} = -\frac{1}{|n_2|} \langle \partial_i r, \partial_j n_2 \rangle.$$

It follows from here

$$c_{11} = c_{21} = c_{21} = 0, c_{22} = -\frac{\sqrt{|c_2|}}{ch2v}$$

Let given $x(x_1, x_2)$ point and $t(t_1, t_2)$ vector in (u, v) parametrization. Then normal curvature at the point $x(x_1, x_2)$ in a direction of the vector $t(t_1, t_2)$ is calculated by the formula:

$$k_N(x, t) = \left(\left(\frac{\sum_{i=1}^2 b_{ij} t_i t_j}{\sum_{i=1}^2 g_{ij} t_i t_j} \right)^2 + \left(\frac{\sum_{i=1}^2 c_{ij} t_i t_j}{\sum_{i=1}^2 g_{ij} t_i t_j} \right)^2 \right)^{\frac{1}{2}}$$

From this formula we have that normal curvature at the any point $x(x_1, x_2)$ in any direction $t(t_1, t_2)$ is a positive and equal to:

$$k_N(x, t) = \left(\frac{\frac{|c_1|}{ch^2 2u} t_1^4 + \frac{|c_2|}{ch^2 2v} t_2^4}{|c_1| ch 2u t_1^2 + |c_2| ch 2v t_2^2} \right)^{\frac{1}{2}}$$

Now we calculate normal torsion at the point $x(x_1, x_2)$ in the direction $t(t_1, t_2)$ by the formula from and we have following:

$$\sigma_N(x, t) = (|c_1| ch 2u t_1^2 + |c_2| ch 2v t_2^2)^{\frac{1}{2}} \left(\frac{\frac{4\sqrt{|c_2|} sh 2v}{ch^2 2v} t_1^2 t_2^3 + \frac{4\sqrt{|c_1|} sh 2u}{ch^2 2u} t_1^3 t_2^2}{\frac{\sqrt{|c_1|}}{ch^2 2u} t_1^2 + \frac{\sqrt{|c_2|}}{ch^2 2v} t_2^2} \right)$$

It is to check normal torsion is not equal zero. **Theorem 1 is proved.**

Remark 1. It is known that Gauss curvature of two-dimensional surface in E^4 is calculated by the formula [5,8,9]:

$$K_G = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

It follows that $K_G=0$.

Remark 2. It is known that Gauss torsion of two-dimensional surface in E^4 is calculate by the formula [5,8,9]:

$$\sigma_G = \frac{\langle n_{1u}, n_{2v} \rangle - \langle n_{2u}, n_{1v} \rangle}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

From this formula we have $\sigma_G = 0$

Remark 3. The vector field X_1 is a Hamiltonian vector field with the Hamiltonian $H : M^4 \rightarrow R$ which is given by the formula

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 - q_2^2)$$

The vector field X_2 is a Hamiltonian vector field with the Hamiltonian $H : M^4 \rightarrow R$ which is given by the formula

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 - q_1^2)$$

In fact the vector field X_1 generates Hamiltonian system which is completely integrable in the sense of Liouville.

Invariant functions are functionally independent first integrals of this Hamiltonian system and following equations

$$\begin{aligned} F_1 &= p_1^2 - q_1^2 \\ F_2 &= p_2^2 - q_2^2 \end{aligned}$$

generate Liouville foliation.

3 Conclusion

The paper studies the foliation generated by the family of orbits of vector fields. It is shown that regular leaves are two-dimensional manifolds of positive normal curvature and non-zero normal torsion.

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