On the geometry of Liouville foliations

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Abstract. The paper studies the geometry of Liouville foliation generated by integrable Hamiltonian system. It is shown that regular leaves are two-dimensional surface of zero Gaussian curvature and zero Gaussian torsion.

1 Introduction

The basic concept of a Hamiltonian system of differential equations forms the basis of much of the more advanced work in classical mechanics, including motions of rigid bodies, celestial mechanics, quantization theory and so on. More recently, Hamiltonian methods have become increasingly important in the study of the equations of continuum mechanics, including fluids, plasmas and elastic media [1,2].

2 Materials and methods

We are interested geometry of Liouville foliation generated by hamiltonian systems.

In this paper the geometry of Liouville foliation generated by integrable Hamiltonian system is studied.

Definition 1. [1-3]. Let M be a smooth manifold of dimension *m.*

A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F,H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

1. *Bilineality*:

$$\{cF+c'P,H\}=c\{F,H\}+c'\{P,H\},$$
$$\{F,cH+c'P\}=c\{F,H\}+c'\{F,P\}.\ c,c'\in\mathbb{R}$$

(b) *Skew-Symmetry*:

$$\{F,H\}=-\{H,F\}$$

(c) *Jacobi Identity*:

$$\{\{F,H\},P\}+\{\{P,F\},H\}+\{\{H,P\},F\}=0$$

(d) *Leibniz' Rule*:

$$\{F,H\cdot P\}=\{F,H\}\cdot P+H\cdot\{F,P\}$$

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A manifold M with a Poisson bracket is called *Poisson manifold*, the bracket defining *Poisson structure* on M .

Example 1. Let M be the Euclidean space $\mathbb{R}^m$, $m = 2n + l$ with coordinates $(p, q, z) = (p^1, ..., p^n, q^1, ..., q^n, z^1, ..., z^l)$. If $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^{n} \{\frac{\partial H}{\partial p^i} \cdot \frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i} \cdot \frac{\partial F}{\partial p^i}\}$$

We note the particular bracket identities:

$$\{p^i, p^j\} = 0, \{q^i, q^j\} = 0, \{q^i, p^j\} = \delta^i_j.$$

$$\{p^i, z^k\} = \{q^i, z^k\} = \{z^t, z^k\} = 0.$$

in which i and j run from 1 to n, when t and k run from 1 to l. δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 2. Let M be a Poisson manifold and *H:M*→ $\mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field *sgradH* on M satisfying

$$sgradH(F) = \{F, H\} = -\{H, F\}$$

for every smooth function *F:M*→ $\mathbb{R}$

The equations governing the flow of *sgradH* are referred to as *Hamilton's equations* for the *Hamiltonian function H*.

In the case of the Poisson bracket on *H:M*→ $\mathbb{R}^m$ $m = 2n + l$, the Hamiltonian vector field corresponding to $H(p, q, z)$ is clearly

$$sgradH = \sum_{i=1}^{n} (\frac{\partial H}{\partial p^i} \cdot \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \cdot \frac{\partial}{\partial p^i})$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, i = 1, ..., n \ \frac{dz^j}{dt} = 0, j = 1, ..., l$$

Which are *Hamiltonian systems* in this case. [1-3].

Proposition 1. Let M be a Poisson manifold and *F, H :M* → $\mathbb{R}$ be smooth functions with corresponding Hamiltonian vector fields *sgradF, sgradH*. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:

$$sgrad\{F, H\} = [sgradF, sgradH]$$

Definition 3. Let M^{2n} be a Poisson manifold and *sgradH* Hamiltonian vector field with a smooth Hamiltonian function H.

Hamiltonian system *sgradH* is called *completely integrable in the sense of Liouville*, if exists set of smooth functions $f_1, ..., f_n$ as:

1) $f_1, ..., f_n$ are first integrals of *sgradH* Hamiltonian vector field,

2) they are functionally independent on M , that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields *sgradf*$_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line [1].

Definition 4. Partition of the manifold M^{2n} into connected components of joint level surfaces of the integrals $f_1,...,f_n$ is called *The Liouville foliation* corresponding to the completely integrated system.

Since $f_1,...,f_n$ is preserved by *sgradH,* each leaf of the Liouville foliation is invariant surface. Liouville foliation is consists of regular leaves (which fill almost all *M*) and special leaves (a subset of zero measure) [2].

Let M^{2n} is symplectic manifold with the integrable Hamiltonian vector field *sgradH* in sense of Liouville and its $f_1,...,f_n$ independent involutive integrals. A smooth function

$$\mathscr{F}: M^{2n} \to \mathbb{R}^n, \mathscr{F}(x) = (f_1(x), \dots, f_n(x))$$

is called *Moment display*.

Definition 5. Point x is called *critical point of moment display* if rang $d\mathrm{F}\ (x)$ is less then n. Its image $\mathrm{F}\ (x)$ in $\to \mathbb{R}^n$ is called *critical value*, the set of regular points is called *camera*.

We denote by K the set of critical points of moment display on *M*. Image of K under moment display $\mathrm{F}\,(\mathrm{K}) \subset \mathbb{R}^n$ is called *bifurcation diagram* [2, 4, 5].

3 On the geometry of Liouville foliation

Let us consider the Hamiltonian $H: M^4 \to \mathbb{R}$ on the Poisson manifold M^4 which is given by the formula

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 + q_2^2).$$

The Hamiltonian vector field corresponding to *H* is

$$sgradH = q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2} + p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2}$$

where Hamiltonian system have following form

$$\begin{cases} p_1' = q_1 \\ p_2' = -q_2 \\ q_1' = p_1 \\ q_2' = p_2 \end{cases}$$

By these differential equations we will find smooth functions which are first integrals of this system equations:

$$\begin{cases} p_1^2 - q_1^2 = c_1 \\ p_2^2 + q_2^2 = c_2 \end{cases}$$

Thus we have two functionally independent first integrals of the Hamiltonian system

$$F^1 = p_1^2 - q_1^2 - c_1$$
$$F^2 = p_2^2 + q_2^2 - c_2$$

We can check Poisson brackets of all pair wise combinations F^1, F^2 and H are equal to zero:

$$\{F^1, F^2\} = 0, \{F^1, H\} = 0, \{F^2, H\} = 0.$$

In this case F moment display is function as

$$\mathrm{F}\ (x) = (p_1^2 - q_1^2, p_2^2 + q_2^2)$$

In our case,

$$dF^1 = \{2p_1, -2q_1, 0, 0\}$$
$$dF^2 = \{0, 0, 2p_2, 2q_2\}$$

Set of critical points is $K = K_1 \bigcup K_2$, where

$$K_1 = \{p_1 = 0, q_1 = 0\}$$
$$K_2 = \{p_2 = 0, q_2 = 0\}$$

The bifurcation diagram $\mathrm{F}\ (\mathrm{K})$ is consists of two rays, where it breaks image of moment display to two cameras:

$$\mathrm{F}\ (K_1) = \{0, p_2^2 + q_2^2\}$$
$$\mathrm{F}\ (K_2) = \{p_1^2 - q_1^2, 0\}$$

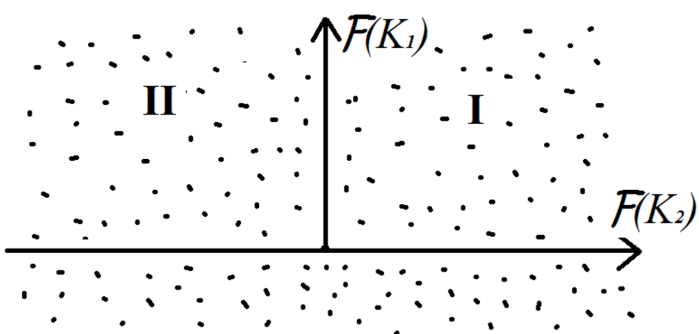

Fig. 1. The camera points marked I and II weighs the regular surfaces of the Liouville foliation.

Now we can check metric characteristics of the surface

$$\begin{cases} p_1^2 - q_1^2 = c_1 \\ p_2^2 + q_2^2 = c_2 \end{cases}$$

Geometry of two dimensional surfaces in the four dimensional Euclidean space is studied by many authors [6].

We will parameterize the surface by the following equations

$$\begin{cases} p_1 = \sqrt{|c_1|} chu \\ p_2 = \sqrt{c_2} cosv \\ q_1 = \sqrt{|c_1|} shu \\ q_2 = \sqrt{c_2} sinv \end{cases}$$

Now we find

$$\frac{\partial r}{\partial u} = r_1 = \{\sqrt{|c_1|} shu; 0; \sqrt{|c_1|} chu; 0\},$$

$$\frac{\partial r}{\partial v} = r_2 = \{0; -\sqrt{c_2} sinv; 0; \sqrt{c_2} cosv\}$$

and coefficients of first quadratic form

$$g_{11} = |c_1| ch2u, \qquad g_{12} = g_{21} = 0, \; g_{22} = \langle r_2, r_2 \rangle = c_2$$

We need two normal vectors to find coefficients of second quadratic forms

$$n_1 = \{0; cosv; 0; sinv\} \qquad \text{and}$$

$$n_2 = \{chu; 0; -shu; 0\}.$$

Now we can find two second quadratic forms of the regular surface. Coefficients of the first of them is calculated by the formula

$$b_{ij} = -\frac{1}{|n_1|} \langle \partial_i r, \partial_j n_1 \rangle.$$

By using these equation we find that

$$b_{11} = b_{12} = b_{21} = 0, \qquad b_{22} = -\sqrt{c_2}$$

Coefficients of the second of them are calculated by the formula

$$c_{ij} = -\frac{1}{|n_2|} \langle \partial_i r, \partial_j n_2 \rangle.$$

It follows from here

$$c_{11} = -\frac{1}{ch2u} \sqrt{|c_1|}, \qquad c_{12} = c_{21} = c_{22} = 0.$$

It is known that Gauss curvature of two-dimensional surface in E^4 is calculated by the formula [4,5,7]:

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

From this formula we have

$$K = \frac{0 \cdot (-\sqrt{c_2}) - 0}{c_2 \,|\, c_1 \,|\, ch2u} + \frac{-\frac{1}{ch2u} \cdot \sqrt{|\, c_1 \,|} \cdot 0 - 0}{c_2 \,|\, c_1 \,|\, ch2u} = 0.$$

It follows that Gauss curvature is a zero.

Now we calculate Gauss torsion of the regular leaves of the Liouville foliation by using following equations

$$F^1 = p_1^2 - q_1^2 - c_1$$
$$F^2 = p_2^2 + q_2^2 - c_2.$$

In this case there is the convenient formula in [6] which we will use.

First of all we need to introduce some denotations.

Let us denote by $\mathbf{h}$ the vector of the dimension 10 with components h_{11}, h_{12}, h_{13}, h_{14}, h_{22}, h_{23}, h_{24}, h_{33}, h_{34}, h_{44}, which are calculated by following formulas

$$h_{ir} = \delta_{ir} - \frac{1}{\Delta}\langle \eta_i, \eta_r \rangle$$

where δ_{ir} is Kronecker symbol, bracket $\langle \cdot, \cdot \rangle$ is inner product and

$$\Delta = \langle gradF^1, gradF^1 \rangle \langle gradF^2, gradF^2 \rangle - \langle gradF^1, gradF^2 \rangle^2.$$

$$\eta_i = (F_i^1 gradF^2 - F_i^2 gradF^1)$$

We also use the denotation $\frac{\partial F^i}{\partial x_k} = F_k^i$ for $p_1 = x_1$, $q_1 = x_2$, $p_2 = x_3$, $q_2 = x_4$ and (6×1) matrix $\mathbf{s}$ with components s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}, where

$$s_{ij} = \varepsilon^{ijkl} \frac{1}{\sqrt{\Delta}} \begin{vmatrix} F_k^1 & F_l^1 \\ F_k^2 & F_l^2 \end{vmatrix}$$

where ε^{ijkl} is Kronecker symbol.

We also introduce (10×6) matrix $\mathbf{B}$

$$\mathbf{B}=\frac{1}{\sqrt{\Delta}}\begin{pmatrix}(1112) & (1113) & (1114) & (1123) & (1124) & (1134)\\ (1212) & (1213) & (1214) & (1223) & (1224) & (1234)\\ (1312) & (1313) & (1314) & (1323) & (1324) & (1334)\\ (1412) & (1413) & (1414) & (1423) & (1424) & (1434)\\ (2212) & (2213) & (2214) & (2223) & (2224) & (2234)\\ (2312) & (2313) & (2314) & (2323) & (2324) & (2334)\\ (2412) & (2413) & (2414) & (2423) & (2424) & (2434)\\ (3312) & (3313) & (3314) & (3323) & (3324) & (3334)\\ (3412) & (3413) & (3414) & (3423) & (3424) & (3434)\\ (4412) & (4413) & (4414) & (4423) & (4424) & (4434)\end{pmatrix}$$

with elements

$$(ijkl)=\begin{vmatrix} F_{ik}^1 & F_{il}^1 \\ F_{jk}^2 & F_{jl}^2 \end{vmatrix}+\begin{vmatrix} F_{jk}^1 & F_{jl}^1 \\ F_{ik}^2 & F_{il}^2 \end{vmatrix}$$

Now we ready to write the formula for the Gauss torsion

$$\boldsymbol{\sigma}_{\mathbf{G}}=\langle \mathbf{h},\mathbf{Bs}\rangle.$$

Let's calculate the required values:

$$\eta_1=(0,4p_1p_2,0,4p_1q_2),\qquad \eta_2=(-4p_1p_2,0,4p_2q_1,0)$$
$$\eta_3=(0,-4p_2q_1,0,-4q_1q_2),\qquad \eta_4=(4p_1q_2,0,-4q_1q_2,0)$$
$$\langle\eta_1,\eta_2\rangle=0,\quad \langle\eta_1,\eta_4\rangle=0,\quad \langle\eta_2,\eta_3\rangle=0,\quad \langle\eta_3,\eta_4\rangle=0$$
$$\langle\eta_1,\eta_1\rangle=4p_1^2\cdot(4p_2^2+q_2^2),\qquad \langle\eta_1,\eta_3\rangle=-4p_1q_1\cdot(4p_2^2+q_2^2),$$
$$\langle\eta_2,\eta_2\rangle=4p_2^2\cdot(4p_1^2+q_1^2),\qquad \langle\eta_2,\eta_4\rangle=-4p_2q_2\cdot(4p_1^2+q_1^2),$$
$$\langle\eta_3,\eta_3\rangle=4q_1^2\cdot(4p_2^2+q_2^2),\qquad \langle\eta_4,\eta_4\rangle=4q_2^2\cdot(4p_1^2+q_1^2),\$$$
$$\Delta=(4p_1^2+4q_1^2)(4p_2^2+4q_2^2)$$
$$h_{11}=\frac{1}{\Delta}\frac{q_1^2}{p_1^2+q_1^2},h_{12}=0,\qquad h_{13}=\frac{1}{\Delta}\frac{p_1q_1}{p_1^2+q_1^2},h_{14}=0,$$
$$h_{22}=\frac{1}{\Delta}\frac{q_2^2}{p_2^2+q_2^2},\qquad h_{23}=0,\qquad h_{24}=\frac{1}{\Delta}\frac{p_2q_2}{p_2^2+q_2^2},$$
$$h_{33}=\frac{1}{\Delta}\frac{p_1^2}{p_1^2+q_1^2},\qquad h_{34}=0,\qquad h_{44}=\frac{1}{\Delta}\frac{p_2^2}{p_2^2+q_2^2}$$
$$s_{12}=\frac{-4q_1q_2}{\sqrt{\Delta}},\quad s_{13}=0,\ s_{14}=\frac{4p_2q_1}{\sqrt{\Delta}},\quad s_{23}=\frac{4p_1q_2}{\sqrt{\Delta}},\quad s_{24}=0,\ s_{34}=\frac{4p_1p_2}{\sqrt{\Delta}}$$

We found $\mathbf{h}$ matrix

$$\mathbf{h}=\frac{1}{\Delta}\begin{pmatrix}\frac{q_1^2}{p_1^2+q_1^2} & 0 & \frac{p_1q_1}{p_1^2+q_1^2} & 0 & \frac{q_2^2}{p_2^2+q_2^2} & 0 & \frac{p_2q_2}{p_2^2+q_2^2} & \frac{p_1^2}{p_1^2+q_1^2} & 0 & \frac{p_2^2}{p_2^2+q_2^2}\end{pmatrix}$$

and $\mathbf{s}$ matrix

$$\mathbf{s}=\frac{4}{\sqrt{\Delta}}\begin{pmatrix}-q_1q_2\\ 0\\ p_2q_1\\ p_1q_2\\ 0\\ p_1p_2\end{pmatrix}$$

Finally, by calculating F_{kl}^{i} coefficients where $k,l=\overline{1,4}\;\; i=1,2$ and all $(ijkl)$ components we got $\mathbf{B}$ matrix

$$\mathbf{B}=\frac{1}{\sqrt{\Delta}}\begin{pmatrix}0&0&0&0&0&0\\4&0&0&0&0&0\\0&0&0&0&0&0\\0&0&4&0&0&0\\0&0&0&0&0&0\\0&0&0&4&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-4\\0&0&0&0&0&0\end{pmatrix}$$

$$\boldsymbol{\sigma}_{\mathbf{G}}=\langle\mathbf{h},\mathbf{Bs}\rangle=0$$

4 Results

Thus we have proved the following theorem:
Theorem 1. Regular leaves of Liouville foliation generated by given Hamiltonian system are two dimensional surfaces of zero Gauss curvature and zero Gauss torsion.

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