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Geometry of some completely integrable Hamiltonian systems

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Abstract. The paper studies the geometry of a Liouville foliation generated by a completely integrable Hamiltonian system. It is shown that regular leaves are two dimensional submanifolds with zero Gaussian curvature and zero Gaussian torsion. It is studied a geometry of the distribution which generates orthogonal foliation to the Liouville foliation.

Keywords: Poisson bracket, Hamiltonian system, Liouville foliation, Gauss curvature, Gauss torsion.

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1. Introduction

The basic concept of a Hamiltonian system of differential equations forms the basis of much of the more advanced work in classical mechanics, including motions of rigid bodies, celestial mechanics, quantization theory and so on. More recently, Hamiltonian methods have become increasingly important in the study of the equations of continuum mechanics, including fluids, plasmas and elastic media [1, 6].

We are interested on the geometry of the Liouville foliation generated by Hamiltonian systems.

In this paper the geometry of the Liouville foliation generated by a completely integrable Hamiltonian system is studied.

1.1. Preliminaries.

Definition 1.1. [1]. A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function {F,H} on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) *Bilineality*:

$$\{cF + c^{'}P, H\} = c\{F, H\} + c^{'}\{P, H\},$$

$$\{F, cH + c^{'}P\} = c\{F, H\} + c^{'}\{F, P\}, \quad c, c^{'} \in \mathbb{R};$$

(b) *Skew-Symmetry*:

$$\{F, H\} = -\{H, F\};$$

(c) *Jacobian Identity*:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0;$$

(d) *Leibnitz' Rule*:

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}.$$

(Here P is an arbitrary smooth real-valued function and $\cdot$ denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called as a *Poisson manifold* and the Poisson bracket defines a *Poisson structure* on M.

Example 1.2. Let M be the Euclidean space $\mathbb{R}^m$, $m = 2n + l$ with coordinates
$(p, q, z) = (p^1, ..., p^n, q^1, ..., q^n, z^1, ..., z^l)$. If $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial p^i} \frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial F}{\partial p^i} \right\} \tag{1.1}$$

We note the particular bracket identities:

$$\{p^i, p^j\} = 0, \{q^i, q^j\} = 0, \{q^i, p^j\} = \delta^i_j,$$

$$\{p^i, z^k\} = \{q^i, z^k\} = \{z^t, z^k\} = 0.$$

in which i and j run from 1 to n, when t and k run from 1 to l. Here δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 1.3. Linear space V is called *symplectic*, if there is a non-degenerate skew-symmetric bilinear form ω.

If there was chosen a basis $e_1, ..., e_m$ in V, then the form ω is uniquely defined by its matrix $\Omega = (\omega_{ij})$ where $\omega_{ij} = \omega(e_i, e_j)$.

A differential 2-form ω is called a *symplectic structure* on a smooth manifold M if it satisfies two conditions:

1) ω is closed, that is $d\omega = 0$,

2) ω is non-degenerated at each point of the manifold, i.e., in local coordinates, $det\Omega(x) \neq 0$, where $\Omega(x) = (\omega_{ij}(x))$ is the matrix of this form.

The manifold endowed with a symplectic structure is called a *symplectic manifold.*

All symplectic manifolds are manifolds of even-dimension and orientable.

Definition 1.4. [2, 5]. Let M be a Poisson manifold and $H \colon M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgradH$ on M satisfying

$$sgradH(F) = \{F, H\} \tag{1.2}$$

for every smooth function $F \colon M \to \mathbb{R}$.

The equations governing the flow of $sgradH$ are referred to as *Hamilton's equations* for the *Hamiltonian function* H.

In the case of the Poisson bracket on $\mathbb{R}^m$ $(m = 2n + l)$, the Hamiltonian vector field to any $H(p, q, z)$, as clearly, corresponds

$$sgradH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right) \tag{1.3}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\begin{cases} \dfrac{dp^i}{dt} = -\dfrac{\partial H}{\partial q^i}, \\ \\ \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p^i}, \\ \\ \dfrac{dz^j}{dt} = 0, \end{cases} \tag{1.4}$$

where $i = 1, \dots, n$ and $j = 1, \dots, l$.

The system (1.4) is called a *Hamiltonian system* with Hamiltonian $H(p, q, z)$[2, 1, 4].

The following property of the Poisson bracket is known[2].

Proposition 1.5. *Let M be a Poisson manifold and $F, H \colon M \to \mathbb{R}$ are smooth functions with corresponding Hamiltonian vector fields $sgradF$, $sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:*

$$sgrad\{F, H\} = [sgradH, sgradF].$$

Definition 1.6. The Hamiltonian vector field associated with $H(x)$ has the form

$$sgradH = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \{x^i, x^j\} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i} \right).$$

Let $F(x)$ be a second smooth function. We obtain the basic formula

$$\{F, H\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j} \tag{1.5}$$

for the Poisson bracket.

This basic brackets $A^{ij}(x) = \{x^i, x^j\}$ $i, j = 1, ..., m$ are called the *structure functions* of the Poisson manifold M with respect to the given local coordinates.

A skew-symmetric $m \times m$ matrix $A(x)$ called the *structure matrix* of M.

If the Poisson bracket is non-degenerate ($det(A^{ij}) \neq 0$ everywhere on M), then the Poisson manifold is called a *symplectic manifold*. The symplectic structure in this case has the form $\omega = A_{ij} dx^i \wedge dx^j$ where A_{ij} are components of the matrix inverse to (A^{ij}).

Definition 1.7. [3]. Let M^m (where $m = 2n$) be a symplectic manifold and $sgradH$ the Hamiltonian vector field with a smooth Hamiltonian function H.

A Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville or completely integrable*, if there exists a set of smooth functions $f_1, \dots, f_n$ such that:

1) $f_1, ..., f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is a natural parameter on their integral trajectories is defined on the whole number line.

Definition 1.8. [2]. A partition of the manifold M^m (where $m = 2n$) into connected components of joint level surfaces of the integrals $f_1, ..., f_n$ is called *the Liouville foliation* corresponding to the completely integrated system.

Since the system $f_1, \dots, f_n$ is preserved by $sgradH$, each leaf of the Liouville foliation is an invariant surface. Any Liouville foliation consists of regular leaves (which fill almost all M) and singular leaves (a set with zero measure).

Let M^{2n} be a Poisson manifold with the integrable Hamiltonian vector field $sgradH$ in sense of Liouville and $f_1, \dots, f_n$ be its independent first integrals.

Let us recall some notions on the geometry of two dimensional submanifolds of four dimensional Euclidean space.

Let two dimensional surface F in $\mathbb{R}^4$ is given with vector function $\mathbf{r} = \mathbf{r}(u, v) \in C^2$.

The *Gaussian curvature* of two dimensional surface F is given by the formula [7]

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}. \tag{1.6}$$

where g_{ij} are coefficients of the first quadratic form, b_{ij} are coefficients of the second quadratic form in the direction of the first normal and c_{ij} are coefficients of the second quadratic form in the direction of the second normal.

The *Gaussian torsion* of two dimensional surface F in $\mathbb{R}^4$ is a function that assigns a value to each point of the surface, calculated by the following formula [8]:

$$\sigma_{\mathbf{G}} = \frac{\sum\limits_{i,j=1}^{2}(b_{i1}c_{j2} - b_{i2}c_{j1})g_{ij}}{\sqrt{g_{11}g_{22} - g_{12}^2}}. \tag{1.7}$$

2. Gaussian curvature and Gaussian torsion of regular leaves

Let $sgradH$ be a completely integrable Hamiltonian vector field with the Hamiltonian function $H\colon \mathbb{R}^4 \to \mathbb{R}$ on the four dimensional Euclidean space with the Cartesian coordinates (p_1, p_2, q_1, q_2)

$$H = H(p_1, p_2, q_1, q_2). \tag{2.1}$$

The Hamiltonian vector field corresponding to H is

$$sgradH = -\frac{\partial H}{\partial q^1}\cdot\frac{\partial}{\partial p^1} - \frac{\partial H}{\partial q^2}\cdot\frac{\partial}{\partial p^2} + \frac{\partial H}{\partial p^1}\cdot\frac{\partial}{\partial q^1} + \frac{\partial H}{\partial p^2}\cdot\frac{\partial}{\partial q^2}, \tag{2.2}$$

where the Hamiltonian system has the following form

$$\begin{cases} p_1' = -\dfrac{\partial H}{\partial q^1}, \\ p_2' = -\dfrac{\partial H}{\partial q^2}, \\ q_1' = \dfrac{\partial H}{\partial p^1}, \\ q_2' = \dfrac{\partial H}{\partial p^2}. \end{cases} \tag{2.3}$$

We assume that the following functions

$$\begin{aligned} f_1 &= f_1(p_1, q_1), \\ f_2 &= f_2(p_2, q_2) \end{aligned} \tag{2.4}$$

are the first integrals of Hamiltonian system (2.3).

Level surfaces of these first integrals generates a Liouville foliation $\mathcal{F}$.

Theorem 2.1. *Regular leaves of a Liouville foliation $\mathcal{F}$ generated by Hamiltonian system (2.3) are two dimensional submanifolds of four dimensional Euclidean manifold with zero Gauss curvature and zero Gauss torsion.*

Proof. A regular leaf of the Liouville foliation is a two dimensional submanifold with equations:

$$\begin{cases} f_1(p_1, q_1) = c_1, \\ \\ f_2(p_2, q_2) = c_2. \end{cases} \tag{2.5}$$

Now we can check metric characteristics of this two dimensional submanifold.

It can be parameterized as:

$$\begin{cases} p_1 = p_1(u), \\ p_2 = p_2(v), \\ q_1 = q_1(u), \\ q_2 = q_2(v). \end{cases} \tag{2.6}$$

Now we find

$$\frac{\partial r}{\partial u} = r_1 = \{p_1'(u); 0; q_1'(u); 0\}, \quad \frac{\partial r}{\partial v} = r_2 = \{0; p_2'(v); 0; q_2'(v)\}$$

and coefficients of the first quadratic form

$$g_{11} = \langle r_1, r_1 \rangle = p_1'^2(u) + q_1'^2(u),$$

$$g_{12} = \langle r_1, r_2 \rangle = \langle r_2, r_1 \rangle = g_{21} = 0,$$

$$g_{22} = \langle r_2, r_2 \rangle = p_2'^2(v) + q_2'^2(v).$$

To find coefficients of the second quadratic forms we need two normal vectors. We choose them as:

$$n_1 = \{-q_1'(u); 0; p_1'(u); 0\}$$

and

$$n_2 = \{0; -q_2'(v); 0; p_2'(v)\}.$$

Now we can find two second quadratic forms of the regular leaf. Coefficients of the first of them is calculated by the formula

$$b_{ij} = -\frac{1}{|n_1|}\langle \partial_i r, \partial_j n_1 \rangle.$$

By using these equation we find that

$$b_{11} = -\frac{1}{\sqrt{p_1'^2(u)+q_1'^2(u)}}(q_1'(u)p_1''(u) - p_1'(u)q_1''(u)), \qquad b_{12} = b_{21} = b_{22} = 0.$$

Coefficients of the second of them are calculated by the formula

$$c_{ij} = -\frac{1}{|n_2|}\langle \partial_i r, \partial_j n_2 \rangle.$$

From here

$$c_{11} = c_{12} = c_{21} = 0, \qquad c_{22} = -\frac{1}{\sqrt{p_2'^2(v)+q_2'^2(v)}}(q_2'(v)p_2''(v) - p_2'(v)q_2''(v).$$

Now we are ready to calculate the Gaussian curvature

$$K = \frac{(b_{11} \cdot 0 - 0) + (0 \cdot c_{22} - 0)}{(p_1'^2(u) + q_1'^2(u))(p_2'^2(v) + q_2'^2(v))} = 0 \tag{2.7}$$

and the Gaussian torsion

$$\sigma_{\mathbf{G}} = \frac{(b_{11}\cdot 0 - 0)\cdot g_{11} + (b_{11}c_{22} - 0)\cdot 0 + (0\cdot c_{22} - 0)\cdot g_{22}}{\sqrt{(p_1'^2(u) + q_1'^2(u))(p_2'^2(v) + q_2'^2(v))}} = 0 \tag{2.8}$$

of two dimensional sub manifold with equations (2.5).

Theorem 2.1 is proved. □

Example 2.2. Let us consider the Hamiltonian $H:\mathbb{R}^4 \to \mathbb{R}$ on the Euclidean four dimensional space $\mathbb{R}^4$ which is given by the formula

$$H(p_1,p_2,q_1,q_2) = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 + q_2^2). \tag{2.9}$$

The Hamiltonian vector field corresponding to H is

$$sgradH = q_1\frac{\partial}{\partial p_1} - q_2\frac{\partial}{\partial p_2} + p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}.$$

It is not difficult to check corresponding Hamiltonian system

$$\begin{cases} p_1' = q_1, \\ p_2' = -q_2, \\ q_1' = p_1, \\ q_2' = p_2 \end{cases} \tag{2.10}$$

is completely integrable.

We have two functionally independent first integrals of the hamiltonian system

$$\begin{aligned} f_1 &= p_1^2 - q_1^2, \\ f_2 &= p_2^2 + q_2^2. \end{aligned} \tag{2.11}$$

A leaf of the Liouville foliation is given by the following system of equations

$$\begin{cases} p_1^2 - q_1^2 = c_1, \\ p_2^2 + q_2^2 = c_2 \end{cases} \tag{2.12}$$

and it is regular when $c_1 \neq 0$ and $c_2 > 0$.

Comparing with Theorem 2.1 we know that regular leaves of the Liouville foliation are two dimensional submanifolds with zero Gauss curvature and zero Gauss torsion.

Singular two dimensional leaves are surfaces obtained by Cartesian multiplication of circles with intersecting lines, where one dimensional ones are conjugate hyperbolas and intersecting lines. (Figure 1.)

3. Orthogonal foliation of completely integrable Hamiltonian system

Let M be a C^∞ manifold of dimension m,

Definition 3.1. [4]. A *distribution* P on M is a map which assigns to every point $x \in M$ vector subspace $P(x)$ of T_xM.

Every set of smooth vector fields D generates a distribution, where for every point $x \in M$ matches subspace $P(x) \subset T_xM$, that generated by set of vectors $D(x) = \{X(x)\colon X \in D\}$.

The distribution P is called *completely integrable*, if for every $x \in M$ there is a submanifold L_x of the manifold M such, that $T_yL_x = P(y)$ for all $y \in L_x$. The submanifold L_x of M is

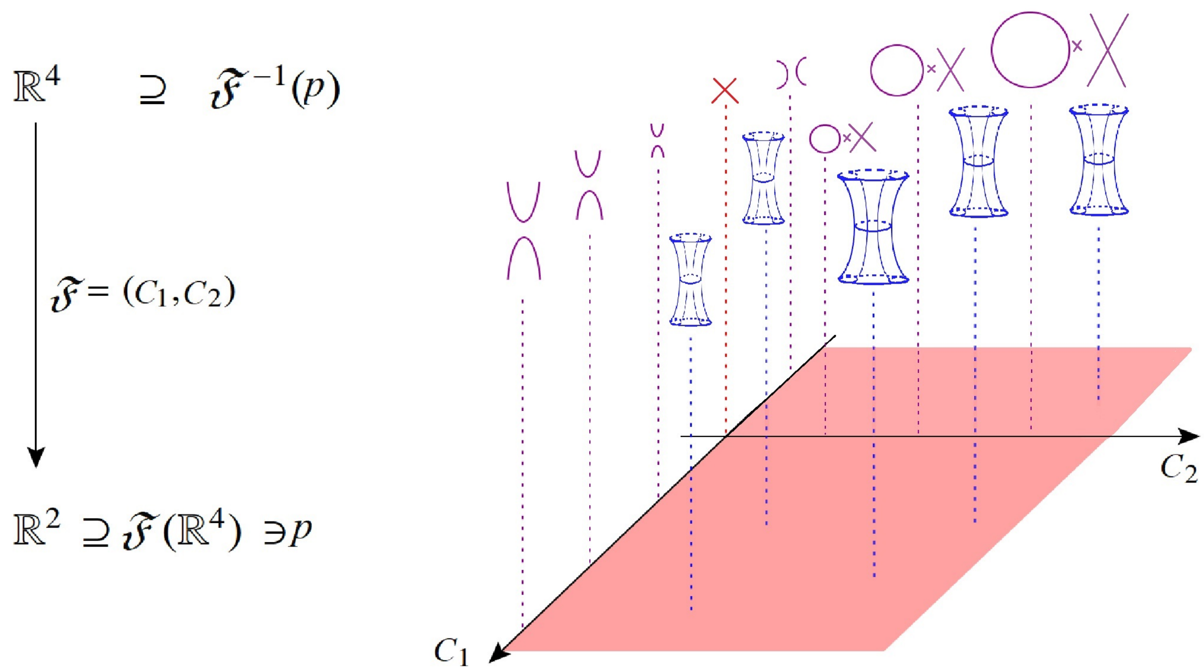

Figure 1. Leaves of the Liouville foliation (2.12)

called an *integral submanifold (or integral manifold)* of the distribution P. A *maximal integral manifold of* P is a connected submanifold L of M such that

(a) L is an integral manifold of P,

(b) every connected integral manifold of P which intersects L is an open submanifold of L.

We say that P is completely integrable if through every point $x \in M$ there passes a maximal integral manifold of P.

Theorem 3.2 (Hermann)**.** [10] *In order a system of smooth vector fields* $D = \{X_1, X_2, ..., X_k\}$ *to generate completely integrable distribution, it is necessary and sufficient that it be involutive.*

Involutiveness of $D = \{X_1, X_2, \dots, X_k\}$ on M means, that for each pair (X_i, X_j) of vector fields there exist smooth real-valued functions $f_{ij}^l(x)$, such that it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^l(x) X_l,$$

$x \in M$, $i, j, l = 1, \dots, k$ (Here $[\cdot, \cdot]$ denotes Lie bracket of smooth vector fields.)

$D \subset V(M)$ be a set of vector fields of all smooth (class C^∞) vector field $V(M)$ and $t \to X^t(x)$ be an integral curve of the vector field X with the initial point x for $t = 0$, which is defined in some region $I(x)$ of real line.

Definition 3.3. [9]. *The orbit* $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, \dots, t_k \in R$ and vector fields $X_1, X_2, \dots, X_k \in D$ such that

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\dots(X_1^{t_1}(x))))$$

where k is an arbitrary positive integer.

The fundamental result in study of orbits is the Sussman theorem.

Theorem 3.4 (Sussman)**.** [9] *Let M be a smooth manifold, and let D be a set of vector fields. Then*

(a) L is an orbit of D, then L admits a unique differentiable structure such that L is a submanifold of M. The dimension of L is equal to its rank.

(b) With the topology and differentiable structure of (a), every orbit of D is a maximal integral submanifold of distribution P.

(c) P has the maximal integral manifolds property.

(d) P is involutive.

Definition 3.5. A partition F of the manifold M by path-connected immersed submanifolds L_α is called *a singular foliation of M* if it verifies condition:

for each leaf L_α and each vector $v \in T_pL_\alpha$ at the point p there is $X \in XF$ such that $X(p) = v$, where T_pL_α is the tangent space of the leaf L_α at the point p, XF is the module of smooth vector fields on M tangent to leaves (XF acts transitively on each leaf).

If the dimension of L is maximal, it is called regular, otherwise L is called singular. It is known that orbits of vector fields generate singular foliation.

Let us denote by P the distribution generated by vector fields

$$\begin{aligned} grad f^1 &= \{p_1'(u); 0; q_1'(u); 0\}, \\ grad f^2 &= \{0, p_2'(v), 0, q_2'(v)\}. \end{aligned} \tag{3.1}$$

Theorem 3.6. *The distribution P generates foliation $\mathcal{F}^\perp$, which is orthogonal to Liouville foliation $\mathcal{F}$ and regular leaves of singular foliation $\mathcal{F}^\perp$ generated by integral submanifolds of P are two dimensional surfaces of zero Gauss curvature and zero Gauss torsion.*

Proof. The system of vector fields $D = \{X_1, X_2\}$, where $X_1 = grad f_1$, $X_2 = grad f_2$ is involutive as Lie bracket of vector fields is

$$[X_1, X_2] = 0.$$

It follows from Sussman theorem the distribution P is completely integrable.

Now we assume a functions

$$\begin{aligned} \mathfrak{F}_1 &= \mathfrak{F}_1(p_1, q_1), \\ \mathfrak{F}_2 &= \mathfrak{F}_2(p_2, q_2) \end{aligned} \tag{3.2}$$

satisfy following conditions

$$\frac{\partial f_1}{\partial p_1}\frac{\partial \mathfrak{F}_1}{\partial p_1} + \frac{\partial f_1}{\partial q_1}\frac{\partial \mathfrak{F}_1}{\partial q_1} = 0, \quad \frac{\partial f_2}{\partial p_2}\frac{\partial \mathfrak{F}_2}{\partial p_2} + \frac{\partial f_2}{\partial q_2}\frac{\partial \mathfrak{F}_2}{\partial q_2} = 0.$$

From this conditions we have got the following equalities

$$grad f_1(\mathfrak{F}_1) = 0, \quad grad f_1(\mathfrak{F}_2) = 0$$

$$grad f_2(\mathfrak{F}_1) = 0, \quad grad f_2(\mathfrak{F}_2) = 0$$

for the functions

$$\mathfrak{F}_1 = \mathfrak{F}_1(p_1, q_1), \quad \mathfrak{F}_2 = \mathfrak{F}_2(p_2, q_2).$$

It follows from those equalities integral submanifolds of the the distribution P are given by equations

$$\begin{cases} \mathfrak{F}_1(p_1, q_1) = s_1, \\ \mathfrak{F}_2(p_2, q_2) = s_2. \end{cases} \tag{3.3}$$

By technics from proof of Theorem 2.1 we have got that regular leaves of $F^{\perp}$ are two dimensional surfaces of zero Gauss curvature and zero Gauss torsion.

Theorem 3.6 is proved. □

Example 3.7. As an example we will take Hamiltonian system (2.9) which is given in the Example 2.2.

It is shown that, functions

$$f_1 = p_1^2 - q_1^2, \quad f_2 = p_2^2 + q_2^2$$

are functionally independent first integrals of the Hamiltonian system (2.9).

In this case, the system of vector fields $D = \{X_1, X_2\}$ consists of vector fields

$$X_1 = gradf_1 = \{p_1, 0, -q_1, 0\},$$

$$X_2 = gradf_2 = \{0, p_2, 0, q_2\}.$$

D is involutive since the Lie bracket of the vector fields is equal to zero.

It follows from the Sussman theorem that the distribution P is completely integrable.

It is easy to check that functions

$$\mathfrak{F}_1(p_1, q_1) = p_1 \cdot q_1, \quad \mathfrak{F}_2(p_2, q_2) = \frac{p_2}{q_2} \tag{3.4}$$

are invariant functions for the system of the vector fields $\{X_1, X_2\}$.

Distribution P generates foliation leaves of which are given by equations

$$\begin{cases} p_1 \cdot q_1 = c_1, \\ \dfrac{p_2}{q_2} = c_2, \end{cases}$$

where c_1, c_2 are real numbers. It follows from Theorem 2.1 that integral submanifolds of P are two-dimensional surfaces of zero Gauss curvature and zero Gauss torsion.

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