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ON THE GEOMETRY OF $\mathbb{SO}(3)$

ABDIGAPPAR NARMANOV and SHOHIDA ERGASHOVA

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The geometry of the rotation group $\mathbb{SO}(3)$ is of interest in many areas of mathematics and mechanics, and many papers are devoted to the study of this group. In the first part of this paper, the geometry of a known submersion on the rotation group with a base on a two-dimensional sphere is studied. It is proved that this submersion is Riemannian and generates a totally geodesic Riemannian foliation. In the second part of the paper, the geometry of the Liouville foliation on the cotangent bundle $T^*\mathbb{SO}(3)$ of the group $\mathbb{SO}(3)$ generated by a completely integrable Hamiltonian system is studied. It is shown that the regular leaf of this foliation is a three-dimensional surface of non-zero normal curvature and zero Gaussian torsion.

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Contents

1. Introduction

Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g, ∇ be the Levi-Civita connection, and $\langle\cdot,\cdot\rangle$ be the inner product defined by the Riemannian metric g.

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We denote by $V(M)$ the set of all smooth vector fields defined on M, through a $[X, Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with a Lie bracket.

Throughout the paper, smoothness means smoothness of a class C^∞.

Definition 1. *Differentiable mapping $\pi\colon M \to B$ of a maximal rank, where B is the smooth manifold of dimension m, $n > m$, called submersion.*

By the theorem of the rank of a differentiable function for each point $p \in B$ the full inverse image $\pi^{-1}(p)$ is a submanifold of M dimension $k = n-m$. Thus submersion $\pi\colon M \to B$ generates a foliation F on M of dimension $k = n - m$, whose leaves are connected components of the submanifolds $L_p = \pi^{-1}(p)$, $p \in B$.

Numerous papers [4, 6, 9, 10, 14, 15] have devoted to the study of the geometry of submersions. In particular, the fundamental equations of submersion have been derived in [14].

Let F be a foliation of dimension k, where $0 < k < n$ [6]. We denote by L_p the leaf of foliation F, passing through a point $q \in M$, where $\pi(q) = p$, by T_qF tangent space of leaf L_p at the point $q \in L_p$, by $H(q)$ orthogonal complement of subspace T_qF. As result arise subbundle's $TF = \{T_qF\}$, $TH = \{H(q)\}$ of the tangent bundle TM and we have an orthogonal decomposition $TM = TF \oplus TH$. Thus every vector field X is decomposable as: $X = X^v + X^h$, where $X^v \in TF$, $X^h \in TH$. If $X^h = 0$ (respectively $X^v = 0$), then the field X is called as vertical (respectively horizontal) vector field.

Definition 2. *The submersion $\pi\colon M \to B$ is said to be Riemannian if differential $d\pi$ preserves the lengths of the horizontal vectors.*

Riemannian submersions are known to generate Riemannian foliations [4, 15].

We note that a foliation F is called Riemannian if every geodesic, orthogonal at some point to the leaves, remains orthogonal to the leaves at all points.

The curve is called horizontal if its tangential vector is horizontal.

Let $\gamma\colon [a, b] \to B$ be a smooth curve in B, and $\gamma(a) = p$. The horizontal curve $\tilde{\gamma}\colon [a, b] \to M$, $\tilde{\gamma}(a) \in \pi^{-1}(p)$ is called the horizontal lift of a curve $\gamma\colon [a, b] \to B$, if $\pi(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$.

The horizontal vector field U is called basic if the vector field $[Y, U]$ is also vertical for each vector field $Y \in V(F)$.

The special orthogonal group $\mathbb{SO}(3)$ consists of all 3×3 orthogonal matrices with determinant one. It is a Lie group, and its associated manifold can be understood in several ways (cf [8]). The group $\mathbb{SO}(3)$ itself is defined as

$$\mathbb{SO}(3) = \{A \in \mathbb{GL}(3, \mathbb{R});\ A^T A = E, \det(A) = 1\}.$$

Here, A^T denotes the transpose of A, and E is the identity matrix.

The set $\mathbb{SO}(3)$ is the intersection of the orthogonal condition (which defines a submanifold) and the determinant condition (which defines another submanifold). The intersection of two smooth submanifolds in $\mathbb{R}^9$ is itself a smooth manifold, provided the gradients of the defining functions are linearly independent at the points at the intersections.

At a point $A \in \mathbb{SO}(3)$:

- The orthogonality condition can be represented as $f(A) = A^T A - E = 0$.
- The determinant condition can be represented as $g(A) = \det(A) - 1 = 0$.

The gradients of these functions with respect to the entries of A can be shown to be linearly independent at the points in $\mathbb{SO}(3)$.

Since $\mathbb{SO}(3)$ is defined as the zero set of smooth functions ($f(A)$ and $g(A)$), and the gradients of these functions are linearly independent, we conclude that $\mathbb{SO}(3)$ is a compact three-dimensional smooth manifold. Specifically, it is obviously a submanifold of $\mathbb{R}^9$.

First, notice that a rotation about the origin in $\mathbb{R}^3$ can be specified by giving a vector for the axis of rotation and an angle of rotation about the axis. We make the convention that the rotation will be counterclockwise for positive angles and clockwise for negative angles when viewed from the tip of the vector of axis rotation. The specification of a rotation by an axis vector and an angle is far from unique.

The rotation determined by the vector v and the angle φ is the same as the rotation determined by the pair $(\lambda v;\ \varphi + 2n\pi)$, where λ is any positive scalar and n is any integer. The pair $(-v;\ \varphi)$ also determines the same rotation. We note that four real numbers are sufficient to specify a rotation – three coordinates for a vector and one real number to give the angle.

As a set (and as a vector space), the set of quaternions is identical to $\mathbb{R}^4$. The vector $(a; x; y; z)$ is written $a + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ when thought of as a quaternion. The number a is called the real part, and the vector part x, y, z.

Quaternions can be multiplied. The multiplication rules are defined by the following relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \qquad \mathbf{ij} = \mathbf{k}, \qquad \mathbf{jk} = \mathbf{i}, \qquad \mathbf{ki} = \mathbf{j}.$$

Reversing the left-right order changes the sign of the product

$$\mathbf{ji} = -\mathbf{k}, \qquad \mathbf{kj} = -\mathbf{i}, \qquad \mathbf{ik} = -\mathbf{j}.$$

The conjugate of a quaternion $r = a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, denoted $\bar{r}$, is defined to be $\bar{r} = a - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. The length or norm of a quaternion r, denoted by $\|r\|$ is its length as a vector in $\mathbb{R}^4$. The formula for the norm of $\|r\| = \sqrt{a^2 + x^2 + y^2 + z^2}$.

Each nonzero quaternion r has a multiplicative inverse, denoted r^{-1}, given by $r^{-1} = \bar{r}/\|r\|$. When r is a unit quaternion, $r^{-1} = \bar{r}$.

The set of all unit quaternions generates the group. It is well known following theorem [3, 5]

Theorem 3. *All the rotations about lines through the origin form a group, homomorphic to the group of all unit quaternions.*

2. Riemannian Submersion on $\mathbb{SO}(3)$

We will prove the following theorem.

Theorem 4. *Let $\pi\colon \mathbb{SO}(3) \to \mathbb{S}^2$ be a map such that $\pi(A) = AN$ for $A \in \mathbb{SO}(3)$, where $N = (0, 0, 1)$. Then*:

a) *there exits a Riemannian metric $\widetilde{g}$ on $\mathbb{S}^2$ such that $\pi\colon \mathbb{SO}(3) \to (\mathbb{S}^2, \widetilde{g})$ is a Riemannian submersion*

b) *foliation generated by π is a totally geodesic foliation*

c) *foliation generated by π is a Riemannian foliation.*

Proof: Here is how a quaternion r determines a linear mapping $A_r\colon \mathbb{R}^3 \to \mathbb{R}^3$. A point $p = (x; y; z)$ in three-space we consider as a quaternion $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ which we will also call pure quaternion. The quaternion product rpr^{-1} is also pure, that is, of the form $x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, and hence can be thought of as a point $(x'; y'; z')$ in three-space. We define the mapping $A_r\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $A_r(x, y, z) = (x'; y'; z')$.

The rotation induced by a unit quaternion $r = a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is given explicitly by the orthogonal matrix

$$\begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - az) & 2(xz + ay) \\ 2(xy + az) & 1 - 2(x^2 + z^2) & 2(yz - ax) \\ 2(xz - ay) & 2(yz + ax) & 1 - 2(x^2 + y^2) \end{pmatrix}.$$

Mapping π in coordinates (a, x, y, z) has the following form

$$\pi(a, x, y, z) = (2(xz + ay), 2(yz - ax), 1 - 2(x^2 + y^2).$$

We show that the rank of the mapping $\pi\colon \mathbb{SO}(3) \to \mathbb{S}^2$ at each point $q = (a, x, y, z)$ is equal to two. The easy calculation shows, that the Jacobi matrix of mapping π has the form:

$$J(\pi) = \begin{pmatrix} \partial\pi/\partial a \\ \partial\pi/\partial x \\ \partial\pi/\partial y \\ \partial\pi/\partial z \end{pmatrix} = \begin{pmatrix} 2y & 2z & 2a & 2x \\ -2x & -2a & 2z & 2y \\ 2a & -2x & -2y & 2z \end{pmatrix}.$$

Since

$$\begin{vmatrix} 2z & -2a \\ 2a & 2z \end{vmatrix}^2 + \begin{vmatrix} 2y & -2x \\ 2x & 2y \end{vmatrix}^2 > 0$$

the rank of the mapping $\pi\colon \mathbb{SO}(3) \to \mathbb{S}^2$ at each point $q = (a, x, y, z)$ is equal to two.

We can also write an explicit formula for the fiber over a point $p(x, y, z) \in \mathbb{S}^2$. For the base point, $p(x, y, z)$, is not the antipode, $(0, 0, -1)$, the quaternion

$$r = \frac{1}{\sqrt{2(1+z)}}(1 + z - \mathbf{i}\, y + \mathbf{j}\, x)$$

will send $(0, 0, 1)$ to (x, y, z). If we put

$$r_t = \cos t + \mathbf{k} \sin t = \mathrm{e}^{kt}$$

the fiber L_p over the point $p(x_0, y_0, z_0)$ is given by quaternions of the form $r\mathrm{e}^{kt}$, $0 \le t \le 2\pi$. In Cartesian coordinates (a, x, y, z) in $\mathbb{R}^4$ the fiber L_p can be parameterized by equations

$$\begin{aligned} a &= \frac{\sqrt{1+z_0}}{\sqrt{2}}\cos(t), & x &= \frac{1}{\sqrt{2(1+z_0)}}(x_0\sin(t) - y_0\cos(t)) \\ z &= \frac{\sqrt{1+z_0}}{\sqrt{2}}\sin(t), & y &= \frac{1}{\sqrt{2(1+z_0)}}(x_0\cos(t) + y_0\sin(t)). \end{aligned} \tag{1}$$

It follows that the fiber L_p is a great circle and that every leaf is a geodesic line. The vector-speed of this curve (a vertical field) has the form

$$X = -z\frac{\partial}{\partial a} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + a\frac{\partial}{\partial z}\cdot$$

This vector field is Killing field. Recall that a vector field X on Riemannian manifold (M, g) is called a Killing vector field if $L_X g = 0$, where $L_X g$ is the Li derivative and g is the Riemannian metric [9]. This condition is equivalent to the condition

$$Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z]) \tag{2}$$

where Y, Z are arbitrary vector fields on (M, g) and $[X, Y]$ is the Lie bracket of the vector fields X, Y [9].

Let X, Y are vector fields on $\mathbb{S}^2$. As $\mathbb{SO}(3)$ is a compact manifold there exists horizontal liftings X^*, Y^* of vector fields X, Y, i.e., there exists horizontal vector fields X^*, Y^* on $\mathbb{SO}3)$ such that $d\pi(X^*) = X$, $d\pi(Y^*) = Y$ (see [6]). Since the vector fields $[X, Y]$ and $[X, Z]$ are vertical vector fields, from (2) we have $Xg(X^*, Y^*) = 0$, i.e., a inner product $g(X^*, Y^*)$ is constant along a fiber of foliation [14]. Hence, if we will put $\langle X, Y\rangle(p) = g(X^*, Y^*)(q)$, where $q \in L_p$, $\langle X,Y\rangle$ is correctly defined inner product on $\mathbb{S}^2$, and we get Riemannian metric $\tilde{g}$ on $\mathbb{S}^2$. Concerning this Riemannian metric submersion $\pi\colon \mathbb{SO}(3) \to \mathbb{S}^2$ will be Riemannian. It is known that a Riemannian submersion generates a Riemannian foliation [4].

Theorem 4 has been proven. □

3. Hamiltonian Systems on $T^*\mathbb{SO}(3)$

Let M^n be a smooth Riemannian manifold with a Riemannian metric $g_{ij}(x)$ of dimension n.

Consider the natural coordinates x and p on the cotangent bundle T^*M, where $x = (x^1, \ldots, x^n)$ are the coordinates of a point on M and $p = (p_1, \ldots, p_n)$ are the coordinates of a covector of the cotangent space $T^*_x M$ on the basis $dx^1, \ldots, dx^n$. Let take the standard symplectic structure $\omega = dx \wedge dp$ on T^*M and consider the following function as a Hamiltonian

$$H(x, p) = \frac{1}{2} \sum g^{ij}(x) p_i p_j = \frac{1}{2} |p|^2. \tag{3}$$

It is well known that the following proposition [2, p. 415].

Proposition 5.

a) *Let $\gamma(t) = (x(t)\,; p(t))$ be an integral trajectory of the Hamiltonian system $v = \operatorname{sgrad} H$ on T^*M. Then the curve $x(t)$ is a geodesic, and its velocity vector $\dot{x}(t)$ is connected to $p(t)$ by the following relation*

$$\frac{dx^i(t)}{dt} = \sum g^{ij}(x) p_j(t).$$

b) *Conversely, if a curve $x(t)$ is a geodesic on M, then the curve $(x(t)\,; p(t))$, where $p_i(t) = \sum g_{ij}(x) p_j(t)$, is an integral trajectory of the Hamiltonian system $v = \operatorname{sgrad} H$.*

Let us consider the Hamiltonian (3) for the manifold $\mathbb{SO}(3)$. To do this, let us parameterize the group $\mathbb{SO}(3)$ by Euler angles. If we write the following replacement of coordinates from (a, x, y, z) in $\mathbb{R}^4$ to (u, v, w) in Euler angles

$$a = \cos\frac{v}{2}\cos\frac{w+u}{2}, \qquad x = \sin\frac{v}{2}\cos\frac{w-u}{2}$$
$$y = \sin\frac{v}{2}\sin\frac{w-u}{2}, \qquad z = \cos\frac{v}{2}\sin\frac{w+u}{2}$$

for $A \in \mathbb{SO}(3)$ we get

$$A = \begin{pmatrix} \mathrm{c}u\,\mathrm{c}w - \mathrm{s}u\,\mathrm{c}v\,\mathrm{s}w & -\mathrm{s}u\,\mathrm{c}w - \mathrm{c}u\,\mathrm{c}v\,\mathrm{s}w & \mathrm{s}v\,\mathrm{s}w \\ \mathrm{c}u\,\mathrm{s}w + \mathrm{s}u\,\mathrm{c}v\,\mathrm{c}w & -\mathrm{s}u\,\mathrm{s}w + \mathrm{c}u\,\mathrm{c}v\,\mathrm{c}w & -\mathrm{s}v\,\mathrm{c}w \\ \mathrm{s}u\,\mathrm{s}v & \mathrm{c}u\,\mathrm{s}v & \mathrm{c}v \end{pmatrix}.$$

We can find tangent vectors or tangent matrices of $\mathbb{SO}(3)$, as

$$\vec{r_u} = \begin{pmatrix} -\mathrm{s}u\,\mathrm{c}w - \mathrm{c}u\,\mathrm{c}v\,\mathrm{s}w & -\mathrm{c}u\,\mathrm{c}w + \mathrm{s}u\,\mathrm{c}v\,\mathrm{s}w & 0 \\ -\mathrm{s}u\,\mathrm{s}w + \mathrm{c}u\,\mathrm{c}v\,\mathrm{c}w & -\mathrm{c}u\,\mathrm{s}w - \mathrm{s}u\,\mathrm{c}v\,\mathrm{c}w & 0 \\ \mathrm{c}u\,\mathrm{s}v & -\mathrm{s}u\,\mathrm{s}v & 0 \end{pmatrix}$$

$$\vec{r_v} = \begin{pmatrix} \mathrm{s}u\,\mathrm{s}v\,\mathrm{s}w & \mathrm{c}u\,\mathrm{s}v\,\mathrm{s}w & \mathrm{c}v\,\mathrm{s}w \\ -\mathrm{s}u\,\mathrm{s}v\,\mathrm{c}w & -\mathrm{c}u\,\mathrm{s}v\,\mathrm{c}w & -\mathrm{c}v\,\mathrm{c}w \\ \mathrm{s}u\,\mathrm{c}v & \mathrm{c}u\,\mathrm{c}v & -\mathrm{s}v \end{pmatrix}$$

and

$$\vec{r_w} = \begin{pmatrix} -\mathrm{c}u\,\mathrm{s}w - \mathrm{s}u\,\mathrm{c}v\,\mathrm{c}w & \mathrm{s}u\,\mathrm{s}w - \mathrm{c}u\,\mathrm{c}v\,\mathrm{c}w & \mathrm{s}v\,\mathrm{c}w \\ \mathrm{c}u\,\mathrm{c}w - \mathrm{s}u\,\mathrm{c}v\,\mathrm{s}w & -\mathrm{s}u\,\mathrm{c}w - \mathrm{c}u\,\mathrm{c}v\,\mathrm{s}w & \mathrm{s}v\,\mathrm{s}w \\ 0 & 0 & 0 \end{pmatrix}$$

where we have made use of the following short notations $\mathrm{c}u = \cos u$, $\mathrm{c}v = \cos v$ and $\mathrm{c}w = \cos w$, and respectively, $\mathrm{s}u = \sin u$, $\mathrm{s}v = \sin v$ and $\mathrm{s}w = \sin w$.

We get the first quadratic form matrix (g_{ij}) and the inverse matrix of the first quadratic form (g^{ij}) as follows

$$(g_{ij}) = \begin{pmatrix} 2 & 0 & 2\sin v \\ 0 & 2 & 0 \\ 2\sin v & 0 & 2 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} \dfrac{1}{2\cos^2 v} & 0 & -\dfrac{\sin v}{2\cos^2 v} \\ 0 & \dfrac{1}{2} & 0 \\ -\dfrac{\sin v}{2\cos^2 v} & 0 & \dfrac{1}{2\cos^2 v} \end{pmatrix}.$$

Consider the natural coordinates x and p on the cotangent bundle $T^*\mathbb{SO}(3)$, where $x = (u, v, w)$ are the coordinates of a point on $\mathbb{SO}(3)$ and $p = (p_1, p_2, p_3)$ are the coordinates of a covector of the cotangent space $T_x^*\mathbb{SO}(3)$ on the basis $\mathrm{d}u$, $\mathrm{d}v$, $\mathrm{d}w$.

In this case the Hamiltonian (3) has the following form

$$H = \frac{1}{2}\Big(\frac{1}{2\cos^2 v}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2\cos^2 v}p_3^2 - \frac{\sin v}{\cos^2 v}p_1 p_3\Big). \tag{4}$$

Hamiltonian system corresponding to the Hamiltonian function (4) is

$$\begin{aligned}
\frac{\mathrm{d}p_1}{\mathrm{d}t} &= 0, & \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{2\cos^2 v}p_1 - \frac{\sin v}{2\cos^2 v}p_3\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{2}p_2, & \frac{\mathrm{d}p_2}{\mathrm{d}t} &= -\frac{\sin v}{2\cos^3 v}(p_1^2 + p_3^2) + \frac{1+\sin^2 v}{2\cos^3 v}p_1 p_3\\
\frac{\mathrm{d}p_3}{\mathrm{d}t} &= 0, & \frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{1}{2\cos^2 v}p_3 - \frac{\sin v}{2\cos^2 v}p_1.
\end{aligned} \tag{5}$$

Let us recall some definitions.

Definition 6. *The Hamiltonian system is called* completely integrable in the sense of Liouville or completely integrable*, if there exists a set of smooth functions* $F_1, \dots, F_n$ *as*

1) $F_1, \dots, F_n$ *are the first integrals of the* sgrad H *Hamiltonian vector field,*

2) *they are functionally independent on* M*, that is, almost everywhere on* M *their gradients are linearly independent.*

3) $\{F_i, F_j\} = 0$ *for any* i *and* j*,*

4) *the vector fields* sgrad F_i *are complete, that is, the natural parameter on their integral trajectories is defined on the whole number line.*

Definition 7. *The decomposition of the manifold* M^{2n} *into connected components of the common-level surfaces of the integrals* $F_1, \dots, F_n$ *is called the* Liouville foliation *corresponding to the integrated system* $v = \operatorname{sgrad} H$*.*

Since $F_1, \dots, F_n$ are preserved by flow of $v = \operatorname{sgrad} H$, every leaf of the Liouville foliation is an invariant surface.

The Liouville foliation consists of regular leaves (filling M almost in the whole) and singular ones (filling a set of the zero measure) [2, 11–13].

Theorem 8. *Hamiltonian system* (5) *defined by* (4) *is completely integrable in the sense of Liouville. Regular leaves of a Liouville foliation generated by the Hamiltonian system* (5) *are three-dimensional submanifolds of the six-dimensional manifold* $T^*\mathbb{SO}(3)$ *with nonzero normal curvature and zero Gaussian torsion.*

Proof: In fact, the following functions are the first integrals of the system

$$F_1 = p_1, \qquad F_2 = H, \qquad F_3 = p_3.$$

It is easy to check that these functions satisfy the conditions of complete integrability.

The leaves of the Liouville foliation generated by the Hamiltonian system (5) are given by the equations

$$p_1 = C_1, \qquad H = C_2, \qquad p_3 = C_3.$$

We can parameterize a leaf F by following parametric equations

$$p_1 = C_1, \qquad p_2 = \sqrt{4C_2 + \frac{2\sin v}{\cos^2 v} C_1 C_3 - \frac{C_1^2 + C_3^2}{\cos^2 v}}, \qquad p_3 = C_3 \qquad (6)$$
$$u = u, \qquad v = v, \qquad w = w.$$

Here, u, v, w are the coordinates on a leaf of the Liouville foliation.

The first quadratic form of the three-dimensional F is the following form

$$I(\tau) = \mathrm{d}u^2 + \left(\left(\frac{\partial p_2}{\partial v}\right)^2 + 1\right)\mathrm{d}v^2 + \mathrm{d}w^2$$

for a tangent vector $\tau = \{\mathrm{d}u, \mathrm{d}v, \mathrm{d}w\}$.

The second quadratic forms in the directions of normals

$$n_1 = \{1, 0, 0, 0, 0, 0\}, \qquad n_2 = \left\{0, -1, 0, 0, \frac{\partial p_2}{\partial v}, 0\right\}, \qquad n_3 = \{0, 0, 1, 0, 0, 0\}$$

are the following forms, respectively

$$\mathrm{II}^1(\tau) = 0, \qquad \mathrm{II}^2(\tau) = -\frac{\partial^2 p_2}{\partial v^2}\,\mathrm{d}v^2, \qquad \mathrm{II}^3(\tau) = 0.$$

Recall the notion of normal curvature vector [1]. Through a point $x_0 \in F^n$ in some tangent direction τ we draw a curve γ lying on F^n. Let s be the length of the arc from γ. Consider the curvature vector k of the curve γ: $k = r_{ss}$. *The normal curvature vector* k_N of the surface F_n in the direction τ at a point x_0 is called a projection of the curvature vector k of the curve γ onto the normal space N_{x_0}.

Let $u^i = u^i(s)$, $i = 1, \dots, n$, be the equation of γ. Then

$$r_{ss} = r_{u^i u^j} \frac{\mathrm{d}u^i}{\mathrm{d}s} \frac{\mathrm{d}u^j}{\mathrm{d}s} + r_{u^i} \frac{\mathrm{d}^2 u^i}{\mathrm{d}s^2}.$$

The projection of the vector k onto the normal space N_{x_0} has the form

$$k_N = \sum_{\sigma=1}^{p}(kn_\sigma)n_\sigma = \sum_{\sigma=1}^{p}(r_{ss}n_\sigma)n_\sigma.$$

Using the expression for r_{ss}, we can write

$$k_N = \sum_{\sigma=1}^{p}(r_{u^iu^j}n_\sigma)\frac{\mathrm{d}u^i}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}n_\sigma = \sum_{\sigma=1}^{p}\frac{\mathrm{II}^\sigma n_\sigma}{\mathrm{I}}.$$

From here it follows that the normal curvature vector does not depend from curve γ. It depends only from direction τ.

Using forms, we can find the normal curvature vector of the leaf [1]

$$k_N = \sum_{\alpha=1}^{3}\frac{\mathrm{II}^\alpha \vec{n_\alpha}}{\mathrm{I}} = \frac{-\dfrac{\partial^2 p_2}{\partial v^2}\,\mathrm{d}v^2}{\mathrm{d}u^2 + \left((\dfrac{\partial p_2}{\partial v})^2 + 1\right)\mathrm{d}v^2 + \mathrm{d}w^2}\vec{n_2}.$$

It follows that normal curvature vector is a nonzero vector.

The *indicatrix of the normal curvature* of a submanifold at a point x_0 is the set of points N_{x_0} of the endpoints of the normal curvature vectors $k_N(\tau)$, taken for each tangent direction τ and plotted from the point x_0.

In our case, the indicatrix of the normal curvature is an ellipsoid. Let us denote by a, b, c the half axis of the *indicatrix of the normal curvature*. By analogy with two-dimensional surfaces in four-dimensional space [1], we define the Gaussian torsion χ_G of a three-dimensional surface in six-dimensional space as

$$\chi_G = 3abc.$$

If a three-dimensional surface in the six-dimensional space lies in some four-dimensional space, then the Gaussian torsion is zero. Indeed, in this case, there are two constant normal vectors, and therefore two second forms of the three quadratic forms are identically zero. Therefore, the ellipsoid of normal curvature becomes a segment. Consequently, the Gaussian torsion is zero.

In our case, the normal vectors n_1, n_3 are constant vectors. If we introduce Cartesian coordinates x, y, z in the normal plane, taking vectors n_1, n_2, n_3 as the basis, the *indicatrix of the normal curvature* is the segment defined by

$$|y| \leq \frac{\left|-\dfrac{\partial^2 p_2}{\partial v^2}\right|\mathrm{d}v^2}{\mathrm{d}u^2 + \left((\dfrac{\partial p_2}{\partial v})^2 + 1\right)\mathrm{d}v^2 + \mathrm{d}w^2}.$$

It follows that Gaussian torsion is zero and Theorem 8 has been proven. □

4. Conclusions

The paper is devoted to the geometry of the rotation group of three-dimensional Euclidean space. The geometry of the rotation group is an important problem in mathematics and in applied problems of mechanics. In the first part of the paper the geometry of the well-known submersion on the rotation group is studied, which arises as a result of the action of this group on the two-dimensional sphere. It is proved that this submersion is Riemannian and generates a totally geodesic Riemannian foliation. In the second part of the paper, the geometry of the Hamiltonian system on the cotangent bundle of the rotation group is studied. It is shown that the regular fiber of this foliation is a three-dimensional surface of non-zero normal curvature and zero Gaussian torsion.

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Currently Abdigappar Narmanov is a professor at the Department of Geometry of the National University of Uzbekistan. He had received his PhD from St. Petersburg University in 1983. The main areas of his interests include foliation theory and its applications, geometry of vector fields, and geometry of the Hamiltonian systems. He can be contacted at the email address: `narmanov@yandex.com`.

Department of Geometry and Topology, National University of Uzbekistan
University Street 4, 100174 Tashkent, UZBEKISTAN

Shohida Ergashova is currently completing his PhD at the National University of Uzbekistan on the topic of Geometry of Hamiltonian Vector Fields. Her area of expertise covers various applications of Hamiltonian vector fields, Symplectic manifolds, as well as differential geometry and their applications in physics. She can be contacted at email address: `shohida.ergashova@mail.ru`.

Department of Geometry and Topology, National University of Uzbekistan
University Street 4, 100174 Tashkent, UZBEKISTAN