On the Geometry of the Orbits of Killing Vector Fields

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Abstract. The study of transformations that preserve the space-time metric plays an extremely important role in mathematical physics. It's sufficient to say that the most important conservation laws are associated with such transformations. These transformations generate the so-called Killing vector field. Killing vector fields in physics indicate the symmetry of the physical model and help find conserved quantities such as energy and momentum. In this paper, we study the question about the classification of the geometry of orbits of Killing vector fields #CSOC1120.

Keywords: Killing Vector Fields · Orbit of the Family of Vector Fields · Foliation

1 Introduction

The geometry of Killing vector fields was studied in the works of W. Killing [1], V. N. Berestovsky [2, 3], Yu. G. Nikonorova [2, 3], M.O. Katanaev [4], A.Narmanov [5, 6] and other authors.

As noted above, in a number of areas of physics, for example, in the theory of the electromagnetic field, in the theory of heat, in static physics and in the theory of optimal control, it is necessary to consider not only vector fields, but a family of vector fields. In this case, the main object of research is the orbit of the system of vector fields.

At present, the geometry of the orbits is one of the important problems of modern geometry, which has been studied by many mathematicians due to its importance in optimal control theory, differential games, and the geometry of singular foliations [5, 7–13].

Definition 1. If the infinitesimal transformations $x \to X^t(x)$ of the field X on the Riemannian manifold M preserves the distance between points X is called a Killing vector field.

Example 1. On the Euclidean space $R^3(x, y, z)$, we have Killing fields:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial z},\quad X_4 = -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y},$$

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$$X_4 = -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y},\ X_5 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z},\ X_6 = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

The infinitesimal transformations of fields $X_1,\ X_2,\ X_3$ are parallel transfers in the direction of the coordinate axes and the fields $X_4,\ X_5,\ X_6$ generate rotations.

Example 2. The Killing vector field on $R^4(x_1, x_2, x_3, x_4)$

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}.$$

is tangent to the three dimensional sphere S^3. The vector lines of this vector field generate a smooth bundle called Hopf bundle.

We need the following proposition [2, 9].

Proposition 1. For a vector field $X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$ the condition $\frac{\partial\, \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0\ i \neq j,\ \frac{\partial \xi_i}{\partial x_i} = 0,\ i = 1, ..., n.$ is necessary and sufficient for it to be a Killing field.

2 The Geometry of Killing Vector Fields

In this section we consider Killing vector fields on two-dimensional surfaces.

Theorem 1. Every Killing vector field on the two-dimensional cylinder is vector field of constant length.

Proof. Suppose the considered surface M is parameterized as follows.

$$\begin{cases} x = Rsinu, \\ y = Rcosu, \\ z = v. \end{cases}$$

The following fields

$$X_1 = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial z}.$$

are Killing vector fields on this manifold.

The vector lines of these fields are, respectively, circles and straight lines, which are parallel to the generatrix of the cylinder. It is known that these lines are geodesic on the cylinder.

For every Killing vector field X on M there are smooth functions $\lambda_1(x, y, z)$ and $\lambda_2(x, y, z)$ such that $X = \lambda_1(x, y, z)X_1 + \lambda_2(x, y, z)X_2$. From proposition 1we obtain the following equality: $\lambda_1(x, y, z) = \lambda_1(z), \quad \lambda_2(x, y, z) = \lambda_2(x, y)$ and

$$\frac{\partial \lambda_1}{\partial z} + y\frac{\partial \lambda_2}{\partial x} = 0, \quad \frac{\partial \lambda_1}{\partial z} - x\frac{\partial \lambda_2}{\partial y} = 0. \tag{1}$$

Lie bracket $[X, X_2]$ has the following form

$$[X, X_2] = \lambda_1[X_1, X_2] + X_2(\lambda_1)X_1 + \lambda_2[X_2, X_2] + X_2(\lambda_2)X_2.$$

A simple calculation shows $[X_1, X_2] = 0$. By $[X_2, X_2] = 0$ we have $[X, X_2] = X_2(\lambda_1)X_1 + X_2(\lambda_2)X_2$ and $\frac{\partial \lambda_1}{\partial z} = 0$. It follows from (1) $\frac{\partial \lambda_2}{\partial x} = \frac{\partial \lambda_2}{\partial y} = 0$. Consequently, $\lambda_1(x, y, z)$ and $\lambda_2(x, y, z)$ are constant.

For a vector line of the field X we have the following system:

$$\begin{cases} \frac{\partial x}{\partial t} = \lambda_1 y \\ \frac{\partial y}{\partial t} = -\lambda_1 x \\ \frac{\partial z}{\partial t} = \lambda_2 \end{cases} \tag{2}$$

The integral lines of the vector field is a helix, if $\lambda_1 \neq 0, \lambda_2 \neq 0$. If $\lambda_1 = 0, \lambda_2 \neq 0$ it is a straight line, if $\lambda_1 \neq 0, \lambda_2 = 0$ it is a circle.

Example 3. Let $M = S^2 \times R^1$ be embedded in R^4 using the following parametric equations

$$\begin{cases} x = \cos u \sin v \\ y = \cos u \cos v \\ z = \sin u \\ w = t. \end{cases}$$

The field $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is the Killing field on the M. The vector lines of the field X passing with starting point (x_0, y_0, z_0, w_0) has the form

$$\begin{cases} x(t) = x_0 cost + y_0 sint \\ y(t) = -x_0 sint + y_0 cost \\ z(t) = z_0 \\ w(t) = w_0 \end{cases}$$

If $z_0 \neq 0$, then the trajectory of this system is not a great circle on S^2, cconsequently it is not a geodesic.

Definition 2. The orbit $L(x)$ of the system D of fields with starting point x is defined (see [8]) as the set of points $y \in M$ for which there exist real numbers $t_1, t_2, \dots, t_k$ and vector fields $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ from D (where k is positive integer) such that

$$y = X_{i_k}^{t_k}(X_{i_k-1}^{t_k-1} \dots (X_{i_1}^{t_1}(x)))$$

Example 4. We consider fields $X = 2x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$ and $Y = 2z\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ on three-dimensional Euclidean space. Their Lie bracket is $[X, Y] = 2x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z}$. The first integral of these vector fields is the surface $y^2 - 2xz = 0$. The orbit of the system D is surfaces that are defined using the equations $y^2 - 2xz = C$, where C is a constant.

3 The Classification of Geometry of Orbits

We have the following classification theorem.

Theorem 2. The geometry of orbits of Killing vector fields on R^3 have one of the following types:

1) All orbits are parallel straight lines;

2) The orbits are concentric circles and centers of these concentric circles.
3) The orbits are helical lines;
4) The orbits are parallel planes;
5) The orbits are concentric spheres and a point;
6) The orbits are concentric cylinders and axis of cylinders;
7) Every orbit coincides with R^3.

For the proof of the Theorem 2 one needs the following lemma from [14].

Lemma. Let F be a partition of the complete Riemannian manifold M into orbits of Killing fields. Let γ_0 is geodesic with starting point x_0 to the point y_0, orthogonal to the orbit. Then for every $x \in L(x_0)$ there $\exists$ a geodesic γ with starting point x to the point of the orbit $L(y_0)$ of length γ_0 and orthogonal to the all orbits.

Proof of the Theorem 2. 1) We assume $L(p_0) = p_0$ for some a unique point p_0. In this case the set D must contain more than one vector field. Let S_r^2 be a sphere with the radius $r > 0$ centered at p_0. Then the infinitesimal transformations of the vector fields from D take the sphere S_r^2 into itself. It follows the orbit $L(q)$ for the point $q \in S_r^2$ is S_r^2. Consequently the orbits are concentric spheres and a point p_0.

2) Let us assume $\exists$ p_1 and p_2 such that $L(p_i) = \{p_i\}$, $i = 1, 2$. Hence it follows that the straight line p_1p_2 is fixed. For other points not on this line, the orbit is a circle with center on this line.
3) Assume $dimL(p) = 1$ for all $p \in R^3$. Then the orbits generate one dimensional foliation.By Lemma leaves of this foliation are helical lines with a common axis or all leaves of the foliation are parallel lines.
4) We assume $dimL(p) \geq 1$ for all points and $dimL(p) = 2$ for some point p and $dimL(q) = 1$ for some point q.

Let $dimL(q) = 1$ for a point q. We have possibilities:

a) the orbit $L(q)$ be a straight line, there are rotations around $L(q)$. It follows the orbits are concentric cylinders with common axis $L(q)$.
b) If orbit $L(q)$ is a helix, then the case is similar to the previous case.

5) If $dimL(p) = 2$ for all points p it follows from [7] that the leaves are parallel planes.
6) There is the point q such that $\dim L(q) = 3$. In this case, from [8] and [9], we have $L(q)$ is a open subset of R^3. It follows from the paper [6] $L(p_0)$ is a closed set. Consequently $L(q) = R^3$ Theorem is proved.

4 On the Compactness of the Orbits

This part of the paper devoted to the geometry of Killing fields under the condition that the vector fields and the Riemannian metric are connected by the condition.

Assume D be a system of smooth vector fields defined on a smooth manifold M of dimension n. Let $A(D)$ denote the minimal Lie subalgebra that contains D, $A_x(D)$ the vector space, consisting of all vectors $\{X(x) : X \in A(D)\}$.

If $dimA_x(D) = k$ for all $x \in M$, where $0 < k < n$, orbits of the system D generate a k dimensional foliation [8]. If we suppose $dimA_x(D) = n - 1$ for all $x \in M$, then the orbits of the system are submanifolds of dimension $n - 1$ [8].

Theorem 3. If $Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z])$ for $X \in A(D)$ and for every fields $Y, Z \in V(M)$ on the complete manifold M, then if some orbit is compact, then all orbits are compact.

Proof. It was proved in [14] that under the condition of the theorem, which connects vector fields and the Riemannian metric g, the orbits of the system D generates a Riemannian foliation of codimension one.

We denote by $F(x)$ the tangent space of the orbit $L(x)$ at the point x, $H(x)$ the orthogonal complement $F(x)$ in T_xM, $x \in M$, where T_xM is the tangent space at x. Two subbundles appear $TF = \{F(x) : x \in M\}$ and $H = \{H(x) : x \in M\}$, where H is the orthogonal complement of TF.

A line $\gamma : [0, 1] \to M$ is called horizontal if $\dot{\gamma}(t) \in H(\gamma(t))$ for each $t \in [0, 1]$. A line that lies in the leaf of the foliation F is called vertical.

We suppose the orbit $L_0 = L(x_0)$ is a compact set. Let L be some orbit other than L_0. For an arbitrary point $x \in L$ by $d(x, L_0)$ we denote the distance from the point x to the orbit L_0. Due to the fact that L_0 is a compact orbit, there is a point $p_x \in L_0$ such that $d(x, L_0) = d(x, p_x)$, where $d(x, p_x)$ is the distance. Since the Riemannian manifold M is complete, there is a geodesic line γ_x that realizes this distance [14].

This means that the geodesic γ_x connects the points x, p_x and its length is equal to the distance $d(x, p_x)$. Note that, in addition, this geodesic is orthogonal to the leaf L_0 [14] and, since the foliation is Riemannian, it is orthogonal to all leaves.

Let $z \in L$ be a point of the orbit other than the point x, $\nu : [0, 1] \to L$ a curve in L connecting the points x and z: $\nu(0) = x$, $\nu(1) = z$. As the foliation F is Riemannian and the M is complete, for each pair of vertical and horizontal lines $\nu, h : I \to M$ with $h(0) = \nu(0)$, there is a piecewise smooth mapping $P : I \times I \to M$ such that the line $t \to P(t, s)$ is a vertical line for every $s \in I$, and the line $s \to P(t, s)$ is horizontal for every $t \in I$, where $P(t, 0) = \nu(t)$ for every $t \in I$ and $P(0, s) = h(s)$ for every $s \in I$. This homotopy is called the vertical-horizontal homotopy.

By the theorem in [8], there exists a homotopy $P : I \times I \to M$ for the curves ν, γ_x such that $P(0, s) = \gamma_x(s)$ for $s \in I$ and for each $t \in I$ the curve $s \to P(t, s)$ is a horizontal geodesic of length $d(x, p_x)$. Thus, the length of the geodesic $\gamma_z(s) = P(1, s)$ is equal to $d(x, p_x)$.

Thus, from each point z of the orbit L comes the geodesic $\gamma_z : [0, 1] \to M$, the length of which is equal to the distance from the point $z \in L$ to L_0, and the lengths of all of geodesics γ_z are equal to $d = d(x, p_x)$.

Now we will show that, the orbit L is compact. Since the orbit L has dimension $n - 1$, only one geodesic passes through each point $z \in L$. Therefore, the geodesic flow $z \to exp_z(v)$ transfers L to L_0, where $v \in T_zM$ is the horizontal vector and $|v| = d$, Since the geodesic smoothly depends on the initial points, this map is bijective, and therefore a diffeomorphism. In particular, as the image of a compact set under a diffeomorphism, the orbit L is compact. The theorem is proved.

5 Applications in Partial Differential Equations

Definition 4. [10] A group G of transformations acting on the set M is called the symmetry group of differential equation if every element $g \in G$ transforms a solution of the equation to a solution.

Let us to bring simple example. For the heat equation, $u_t = u_{xx}$ Killing vector field $X = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$ is infinitesimal generator of symmetry group of the heat equation. The vector field X generates the group of following transformations. We can check that the function $f = bt - ax$ is invariant function of the transformation group. This invariant function allows us to search solution of heat equation in the form $u(t, x) = V(\xi)$, where $\xi = bt - ax$. The function $u(t, x) = V(\xi)$ is a solution of second-order differential equation: $a^2V'' - bV' = 0$. As result, we obtain large class of solutions of the heat equation:

$$u(t, x) = C_1\frac{a}{b}e^{bt-ax} + C_2,$$

where C_1, C_2 are arbitrary constants.

The geometry of Killing fields is used for the study of partial differential equations [10, 15].

6 Conclusion

The topology of the orbits of vector fields on a connected Riemannian manifold is studied when there is a given connection between the vector fields and the Riemannian metric. Classification of the orbits is obtained. In the last section, it is given some applications of the geometry of vector fields in in theory of partial differential equations.

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The Classification of Vegetations Based on Share Reflectance at Spectral Bands

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Abstract. This paper studies the problem of classification for the vegetation based on the share reflectance at spectral bands. For this, the theory and methodology of regression and data analysis, algorithms and technologies of remote sensing, several modern scientific literatures are involved. As a result, a system of regression models was built with factor variables Barley, Corn, and Wheat vegetation in the share reflectance at bands of the spectral space and the parameters based on ordinary least squares method were estimated #CSOC1120.

Keywords: Data Analysis · Remote Sensing · Least Squares Method

1 Introduction

Today, the problem of using remote sensing technology is relevant and a huge amount of scientific literature is devoted to this topic, some of which are given in this study.

Known that [1] precision agriculture is used for mapping, monitoring and analysis to changes in vegetations. In particular, the paper proposes a fast and reliable semi-automated workflow implemented for processing unmanned aerial vehicle multispectral images and aimed at detecting and extracting crowns of olive and citrus trees located in the region of Calabria (Italy) to obtain energy maps within the framework of precision agriculture. Also note that the multi-resolution segmentation task was implemented using layers of spectral and topographic channels, the classification stage was implemented as a process tree.

Further, we note that the work [2] was devoted to the survey of agricultural crops using the technology of remote sensing of the earth. To obtain information about the state of agrobiologically, agrochemical and agrophysical characteristics of crops, IoT sensors and multispectral images of crop areas were used to create agrotechnological maps of crops. We also note that here a deep neural network with two hidden layers was used for data processing.

Also, in work [3], it is intended to determine the hidden dependences arising from the seasonal cycles of the productivity of agricultural crops in the irrigated and rainfed land in the central region of Myanmar, and the system solutions for the assessment and metric measurement of plant cover and vegetation indices of this condition. The results

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