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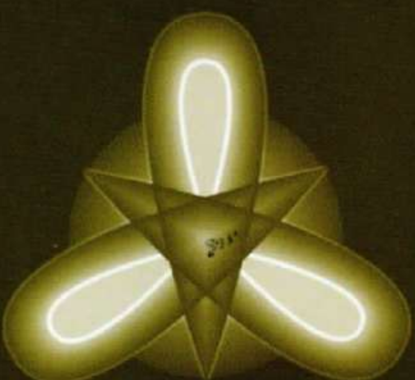

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ON THE GEOMETRY OF SUBMERSIONS

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Abstract. Subject of present paper is the geometry of foliation defined by submersions on complete Riemannian manifold. It is proven foliation defined by Riemannian submersion on the complete manifold of zero sectional curvature is total geodesic foliation with isometric leaves. Also it is shown level surfaces of metric function are conformally equivalent.

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1. Introduction

Differentiable mappings of maximal rank play an important role in all branches of mathematics, in particular in Riemannian geometry. One of the important classes of differentiable mappings of maximal rank consists of immersions. Immersions have been intensively studied since the inception of Riemannian geometry. The geometry of immersed manifolds is a generalization of the classical differential geometry of surfaces in Euclidean space. The differential geometry of isometric immersions and embeddings was well studied and well represented in many textbooks on differential geometry.

The dual concept of submersion was formed relatively recently, in the second half of the twentieth century [2, 14]. The study of the geometry of submersions, in particular the geometry of Riemannian submersions, proved to be very fruitful due to the fact that Riemannian submersions have applications in all sections of modern Riemannian geometry. The study of the geometry of submersions is closely related to the study of the geometry of foliations, which is an important section of modern geometry. Submersion generates a foliation whose leaves are level surfaces of the submersion. Therefore, studying the geometry of submersions is important in the theory of foliations.

Let M be a smooth connected Riemannian manifold of dimension n with Riemannian metric g.

Smoothness in this paper means smoothness of class C^{∞}.

Definition 1. *Differentiable mapping $\pi\colon M \to B$ of maximal rank, where B is a smooth Riemannian manifold of dimension m, called submersion for $n > m$.*

By the rank theorem of a differentiable function the full inverse image $L_p = \pi^{-1}(p)$ of every point $p \in B$ is a submanifold of dimension $k = n - m$. So the submersion of $\pi\colon M \to B$ generates a foliation F of dimension $k = n - m$ on the manifold M, whose leaves are connected components of the inverse images of the points of $p \in B$.

Suppose that L is a leaf of the foliation F, $x \in L$, T_xL is the tangent space of L at the point x, and $H(x)$ is the orthogonal complement of T_xL. There arise two subbundles $TF\colon x \to T_xL$ and $H\colon x \to H(x)$ of the tangent bundle TM of the manifold M. Each vector field $X \in V(M)$ can be represented in the form $X = X_v + X_h$, where X_v and X_h are the orthogonal projections of X onto TF and H, respectively. If $X_h = 0$, then X is called a vertical field (tangent to F), if $X_v = 0$, then X is called a horizontal field.

Geometry of submersions is investigated by many authors, in particularly in papers [1, 5, 15].

Definition 2. *A submersion of $\pi\colon M \to B$ is called Riemannian if its differential $\mathrm{d}\pi$ preserves the length of horizontal vectors.*

To the study of the geometry of Riemannian submersions are devoted many investigations [2, 8, 13, 14]. In particularly fundamental equations of Riemannian submersion are obtained in [13].

2. Geometry of Riemannian Submersions

In this section we study foliations defined by submersions.

Let M be a smooth connected Riemannian manifold of dimension n with Riemannian metric g, and $V(M)$ is the set of all smooth vector fields defined on M. Let us consider submersion

$$\pi\colon M \to B \tag{1}$$

where B is a smooth Riemannian manifold of dimension m and denote by F the foliation defined by submersion.

From a geometrical point of view the important classes of foliations are totally geodesic foliation and Riemannian foliation.

Foliation on a Riemannian manifold is a totally geodesic if every geodesic tange[nt] to the leaf of the foliation at one point lies in this leaf, i.e., each leaf is a total[ly] geodesic submanifold. The geometry of totally geodesic foliation studied in [2,15...

Foliation F is called a Riemannian foliation if each geodesic orthogonal at som[e] point to the leaf of the foliation F remains orthogonal to all leaves F in all the[ir] points [4, p. 189]. Riemannian foliation without singularities were first introduce[d] and studied by Reinhart in [14]. This class of foliations naturally arise in the stud[y] of bundles and geometry of orbits of vector fields [6, 7, 11, 12].

Let us recall notion of C^r-bundle. C^r-submersion $\pi\colon M \to B$, where M is a manifold of dimension n, B is a manifold of dimension m, is called C^r-bundle if there exist an $n - m$ dimensional manifold N and open cover $\{U_\alpha\}$ of B satisfying the following conditions:

1. for every point $p \in B$ submanifold $\pi^{-1}(p)$ is diffeomeorphic to N
2. for each U_α there is a diffeomorphism

$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times N$$

such that

$$\varphi_\alpha(\pi^{-1}(p)) = \{p\} \times N, \qquad p \in U_\alpha.$$

The following theorem holds [2].

Theorem 3. *Let M be a smooth connected complete Riemannian manifold. If (1) is a Riemannian submersion, then it is a bundle. In addition if the foliation F is a totally geodesic, then all leaves are mutually isometric.*

In the case when M is manifold of zero sectional curvature and $B = \mathbb{R}$ Theorem 3 can be strengthened in the following form.

Theorem 4. *Let M be a smooth connected complete Riemannian manifold of zero sectional curvature and*

$$\pi\colon M \to \mathbb{R} \tag{2}$$

is a Riemannian submersion. Then foliation F is a total geodesic Riemannian foliation with isometric leaves.

Proof: Let us recall notion of straight line on the manifold M. Geodesic line $\gamma\colon \mathbb{R} \to M$ is called straight line if $\gamma\colon [a,b] \to M$ is shortest line for every segment $[a,b] \subset \mathbb{R}$. As (2) is a Riemannian submersion it is proven in [9] that every gradient line (integral line of unity gradient vector field Z of the submersion (1)) $\gamma(s)$ is a geodezic line. As M is complete manifold $\gamma(s)$ is defined for all $s \in \mathbb{R}$ and it is diffeomorphic to $\mathbb{R}$ and meets every leaf only one time. We have that gradient lines are parallel straight lines.

Let L be some leaf of F, $v\colon [0,l] \to L$ is the shortest line in M, parameterized with arc length in direction $X_0 \in T_pL$, where $p = v(0)$.

The shortest curve $v\colon [0,l] \to L$ is orthogonal to the gradient line at the point p. As manifold M is manifold of zero sectional curvature and all gradient lines are parallel straight lines, the shortest curve $v\colon [0,l] \to L$ is orthogonal to the all gradient lines as they are parallel [3, Lemma 8, p. 330]). It follows from here shortest line $v\colon [0,l] \to L$ stays in the leaf L. As every geodesic line is a locally shortest line we have that every geodesic which tangents some leaf L of the foliation F stays in this leaf. It follows from here the foliation is total geodesic foliation.

Now we show that the flow of the unity gradient vector field translates leaves of into leaves.

Let $\gamma(t,s)$ be a gradient line starting from the point $v(t)$ at $s=0$: $\gamma(t,0)=v(t)$, $\gamma'(t,0)=Z(v(t))$, where Z is unity gradient vector field of the submersion (1). Let g is the Riemannian metric on M, ∇ is Levi-Civita connection, $\langle U,W\rangle$ is the inner product of vector fields U and W.

As (2) is a Riemannian submersion, the foliation F is a Riemannian [14]. Since F is a Riemannian foliation, it holds following equality [10]

$$X\langle Y,W\rangle = \langle [X,Y],W\rangle + \langle Y,[X,W]\rangle \tag{3}$$

for horizontal vector fields Y, W, where X is a vertical vector field, $[X,Y]$ is the Lie bracket of vector fields X and Y. As it holds $X\langle Z,Z\rangle = 0$ for unity gradient vector field Z we have from (3) the equality

$$\langle [X,Z],Z\rangle = 0 \tag{4}$$

for every vertical vector field X. It follows from here that $[X,Z]$ is a vertical vector field. Therefore Z is a foliated vector field and flow of field Z translates the leaves of F into leaves F [15]. That is why curve $t \to \gamma(t,s)$ lies on the same leaf for $s \in \mathbb{R}$.

Consider two dimensionally surface

$$\Phi = \{\gamma(t,s);\, t\in[0,l_0],\ s\in(-\infty,+\infty)\}.$$

The surface Φ is considered with the restriction $\widetilde{g}$ of Riemannian metric g on M. Restriction $\widetilde{g}$ of Riemannian metric g on Φ gives the metric

$$E(t,s)\mathrm{d}t^2 + \mathrm{d}s^2, \qquad E(t,s) = |X(t,s)|^2$$

where $|X(t,s)|$ is the length of tangent vector $X(t,s)$ of curve $t \to \gamma(t,s)$ [illegible] $p=\gamma(t,s)$.

Since ∇ is Levi-Chivita connection it holds

$$X\langle Z,Z\rangle = \langle \nabla_X Z, Z\rangle + \langle Z, \nabla_X Z\rangle$$

From (5) it follows $\langle \nabla_Z X, Z\rangle = \langle \nabla_X Z, Z\rangle = 0$. Differentiating $\langle X,X\rangle$ [illegible] in the direction of Z and X gives equalities $\langle \nabla_Z X, X\rangle = \langle \nabla_X Z, X\rangle = 0$ [illegible] $\langle \nabla_X X, X\rangle = 0$, respectively. From the equality $\langle X,Z\rangle = 0$ we get $\langle \nabla_X X, Z\rangle$ [illegible]

Thus, $\nabla_X X = 0$ at all points of the surface. This implies that the curves $t \to \gamma(t,s)$ for each s are geodesic line of M.

As manifold M is a manifold of zero sectional curvature it follows that the curvature of the surface Φ is zero [3]. It follows from here that $E(t,s) = |X(t,s)|^2 = 1$, and every curve $t \to \gamma(t,s)$ has length equal to the length of $v\colon [0,l] \to L$. So every geodesic on M is tangent to the leaf of foliation F and stay on this leaf. Besides, as $\nabla_Z X = 0$, the flow of Z sends the geodesics of the leaf to the geodesics the same length. It follows, the foliation F is totally geodesic foliation with isometric leaves. □

Corollary 5. *If the manifold M is simple connected under the condition of Theorem 3 as follows from* [8] *the manifold M is isometric to the product $L\times\mathbb{R}$, where L is a leaf of the foliation F.*

Example 6. Let N is a Riemannian manifold of dimension m, and $M = N\times\mathbb{R}$ is the cartesian product of Riemannian manifolds with product Riemannian metric. If one defines the submersion

$$\pi\colon M \to \mathbb{R} \tag{6}$$

by formula $\pi(u,v) = v$, where $(u,v)\in N\times\mathbb{R}$, then it is a Riemannian submersion. The foliation consists of total geodesic submanifolds $N\times\{v\}$, $v\in\mathbb{R}$.

Example 7. However, for general submersions which are not Riemannian submersions Theorem 4 is not true. For example, if $M=\mathbb{R}^2$ for the submersion $\pi(x_1,x_2) = x_1^2 - x_2$ the foliation F is not total geodesic.

Example 8. Next example shows that in the general case Theorem 4 is not true for Riemannian submersions, too. Any Riemannian submersion can be used to generate new ones by deforming the metric in the vertical direction. To be specific, let

$$\pi\colon (M,g) \to B \tag{7}$$

be a Riemannian submersion. Given function $\varphi\colon M\to\mathbb{R}$, define a new inner product (metric) $\langle\cdot,\cdot\rangle_\varphi$ on M by

$$\langle X,Y\rangle_\varphi = e^{2\varphi(p)}\langle X_v, Y_v\rangle_g + \langle X_h, Y_h\rangle_g$$

where $X,Y\in V(M)$, $p\in M$. Since the horizontal metric is unchanged, π is still a Riemannian submersion.

We now will assume that the manifold M is a Riemannian product $N\times B$, and $\pi\colon M\to B$ is a projection to the second factor. We assume in addition that the function φ depends on points of B only. Then the resulting warped space $(M,\langle\cdot,\cdot\rangle_\varphi)$ is called a warped product and is denoted by $N\times_\varphi B$. If $M = N\times_\varphi\mathbb{R}$, then π is a Riemannian submersion, but the fibers of F are only conformally equivalent.

3. Geometry of Metric Functions

Definition 9. *Differentiable function*

$$f\colon M \to \mathbb{R} \tag{8}$$

is called metric function if length $|\operatorname{grad} f|$ *of gradient vector is constant on every component of level surfaces* [5].

In this section we study metric functions on Euclidean space.

Assumption. Every connected component of the critical level set is either a point either a regular surface (smooth manifold) and they are isolated from each other.

Theorem 10. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is a metric function and assumption holds for it. Then level surfaces of the function* f *are conformally equivalent.*

Note. By conformal equivalence of surfaces we mean there is a conformal map between them. A diffeomorphism between two Riemannian manifolds is called a conformal map if the pulled back metric is conformally equivalent to the original one. Two Riemannian metrics $\widetilde{g}$ and g on a smooth manifold M are called conformally equivalent if $\widetilde{g} = e^{2\psi} g$ for some function ψ on M.

Proof: Under assumption it follows from [5, Theorem 1] that there exists a function $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ such that every level surface $L_c = \{x;\ f(x) = c\}$ of the function f is level surface of the function $\widetilde{f}$ given by equation $\{x;\ \widetilde{f}(x) = \widetilde{c}\}$ and

$$\widetilde{f}(x) = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

where

$$A_1^2 + A_2^2 + \cdots + A_n^2 > 0$$

if the function f has no critical level surfaces of dimension less than $n-1$, and

$$\widetilde{f}(x) = \frac{1}{2}\sum_{i=1}^{n-k}(x_i - a_i)^2$$

where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-k$, if the minimal dimension of critical level surfaces is equal k, where $0 \le k \le n-2$.

Let $Z = \operatorname{grad} \widetilde{f}$ is gradient vector field of the function $\widetilde{f}$. If the function has no critical level surfaces of dimension less than $n-1$, then

$$Z = \operatorname{grad} \widetilde{f} = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}$$

is a constant vector field.

If the function has critical level surfaces of dimension less than $n-1$, then

$$Z = \operatorname{grad} \widetilde{f} = \sum_{i=1}^{n-k}(x_i - a_i)\frac{\partial}{\partial x_i}. \tag{10}$$

Let us recall the notion of conformal vector fields.

Definition 11. *A vector field* X *is conformal if* $L_X g = \sigma g$*, where* σ *is a function on* (M, g)*,* $L_X g$ *denotes the Lie derivative of the metric* g *with respect to* X.

It is known that a vector field X on (M, g) is conformal if and only if the local one-parameter group of local transformations generated by vector field X consists of conformal transformations. A local one-parameter group of local transformations generated by a conformal vector field consists of homotheties if σ is a constant, and consists of isometries if $\sigma = 0$.

It is known that a vector field [6]

$$X = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}$$

on R^n is conformal if and only if

$$\frac{\partial \xi_i}{\partial x_i} = \mu(x), \qquad i = 1, 2, \ldots, n, \qquad \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0, \qquad i \neq j.$$

It is not difficult to check that gradient vector field (8) is conformal vector field. Gradient vector field (9) is conformal vector field on the planes $x_{k+1} = \text{const}$, $x_{k+2} = \text{const}, \ldots, x_n = \text{const}$.

Now we will show that flow of the gradient vector field translates level surface to level surface.

It is known that metric function generates Riemannian foliation F [15]. Since foliation F is Riemannian we have

$$X\langle Z, Z\rangle = \langle [X, Z], Z\rangle + \langle Z, [X, Z]\rangle$$

for each vertical vector field X, where $[X, Z]$ is the Lie bracket of the vector fields X and Z.

It follows from here that $[X, Z]$ is vertical vector field. This is equivalent to that flow of the gradient vector field $Z = \operatorname{grad} \widetilde{f}$ translates level surface of the function $\widetilde{f}$ to level surface $\widetilde{f}$ [15].

As every level surface of the function f is a level surface of the function $\widetilde{f}$ flow of the gradient vector field $Z = \operatorname{grad} \widetilde{f}$ translates level surface of the function f to level surface f. □

Example 12. **A)** Let $f : \mathbb{R}^3 \to \mathbb{R}$ is given by formula

$$f(x_1, x_2, x_3) = \sin(x_1^2 + x_2^2 + x_3^2)$$

where (x_1, x_2, x_3) are the Cartesian coordinates on $\mathbb{R}^3$.
It is easy to check that the function f is a metric function. Critical level surfaces of f are given by equations

$$x_1^2 + x_2^2 + x_3^2 = 0, \qquad x_1^2 + x_2^2 + x_3^2 = \frac{\pi}{2} + l\pi, \qquad l \in \mathbb{N}.$$

Minimal dimension of critical surfaces is $k = 0$. Therefore

$$\widetilde{f}(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{2}$$

and

$$Z = \operatorname{grad} \widetilde{f} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

The vector field $Z = \operatorname{grad} \widetilde{f}$ is conformal on $\mathbb{R}^3$ and its flow translates level surfaces to level surface. Thus level surfaces of the function f are conformally equivalent.
For conformally equivalent metrics $\tilde{g} = e^{2\psi} g$ sectional curvature in the two-dimensional direction defined by an orthonormal pair of vectors X and Y are related as follows

$$\tilde{K}_{X,Y} = h^2 K_{X,Y} + h[\mathrm{Hess}_h(X,X) + \mathrm{Hess}_h(Y,Y)] - |\nabla h|^2$$

where $h = e^{-\psi}$, ∇h, Hess_h are gradient vector and hessian of the function h, respectively. The flow of the vector field $Z = \operatorname{grad} \widetilde{f}$ consists of homoteties

$$Z^t : (x_1, x_2, x_3) \to (x_1 e^t, x_2 e^t, x_3 e^t), \quad t \in \mathbb{R}.$$

For this example

$$h = e^t, \qquad \mathrm{Hess}_h(X,X) = \mathrm{Hess}_h(Y,Y) = 0, \qquad |\nabla h|^2 = 0.$$

Thus we have following relation for section curvatures (Gauss curvatures) for level surfaces $Z^t(L)$ and L

$$\frac{\tilde{K}_{X,Y}}{K_{X,Y}} = e^{2t}.$$

Vector fields

$$V_1 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}, \qquad V_2 = \frac{\partial}{\partial x_3}$$

are vertical vector fields. Their flows consists of motions of Euclidean space. Therefore main curvatures of level surfaces are invariant functions of the group of this transformations.

B) Let $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x_1, x_2, x_3) = \sin(x_1^2 + x_2^2)$$

where (x_1, x_2, x_3) are the Cartesian coordinates on $\mathbb{R}^3$.
It is easy to check that the function f is a metric function. Critical level surfaces of f are given by equations

$$x_1^2 + x_2^2 = 0, \qquad x_1^2 + x_2^2 = \frac{\pi}{2} + l\pi, \qquad l \in \mathbb{N}.$$

the minimal dimension of the critical surfaces is $k = 1$. Therefore

$$\widetilde{f}(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2}{2}$$

and

$$Z = \operatorname{grad} \widetilde{f} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$

The vector field $Z = \operatorname{grad} \widetilde{f}$ is conformal on plane $x_3 = \mathrm{const}$ and its flow translates level surfaces to level surface. Thus level surfaces of the function f are conformally equivalent.
C) Let $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x_1, x_2, x_3) = \sin(x_3^2)$$

where (x_1, x_2, x_3) are the Cartesian coordinates on $\mathbb{R}^3$.
It is easy to check that the function f is a metric function. Critical level surfaces of f are given by equations

$$x_3 = 0, \qquad x_3^2 = \frac{\pi}{2} + l\pi, \qquad l \in \mathbb{N}.$$

The function f has critical level surfaces of dimension $n - 1 = 2$. Therefore

$$\widetilde{f}(x_1, x_2, x_3) = A x_3$$

and

$$Z = \operatorname{grad} \widetilde{f} = A \frac{\partial}{\partial x_3}.$$

The vector field $Z = \operatorname{grad} \widetilde{f}$ is conformal on $\mathbb{R}^3$ and its flow translates level surfaces to level surface. Thus level surfaces of the function f are conformally equivalent.

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References

[1] Abdishukurova G., Narmanov A. and Sharipov X., *Differential Invariants of Parametrical Group of Transformations*, Math. Stat. **8** (2020) 347–352.

[2] Hermann R., *A Sufficient Condition that a Mapping of Riemannian Manifolds ... Fiber Bundle*, Proc. Amer. Math. Soc. **11** (1960) 236–242.

[3] Gromoll D., Meyer W. and Klingenberg W., *Riemannsche Geometrie im ...*, Springer, Berlin 1975.

[4] Molino P., *Riemannian Foliations*, Burkhäuser, Boston 1988.

[5] Narmanov A. and Kaipnazarova G., *Topology of Foliations Generated by ... faces*, Uzbek. Math. J. **2** (2008) 53–60.

[6] Narmanov A. and Rajabov E., *On the Geometry of Orbits of Conformal ...*, J. Geom. Symmetry Phys. **51** (2019) 29–39.

[7] Narmanov A. and Qosimov O., *On the Geometry of the Set of Orbits of Killing ... Fields on Euclidean Space*, J. Geom. Symmetry Phys. **55** (2020) 39–49.

[8] Narmanov A., *On the Geometry of Totally Geodesic Riemannian Foliations* ... Adv. Math. **10** (2000) 104–111.

[9] Narmanov A. and Kaipnazarova G., *Metric Functions on Riemannian ...*, Uzbek Math. J. **2** (2010) 113–120.

[10] Narmanov A., *On the Transversal Structure of the Controllability Set of ... Control Systems*, Differ. Equ. **32** (1996) 607–634.

[11] Narmanov A., *On the Dependence of the Controllability Set on the ...*, Differ. Equ. **33** (1997) 1341–1345.

[12] Narmanov A. and Saitova S., *On the Geometry of Orbits of Killing ...*, Differ. Equ. **50** (2014) 1584–1591.

[13] O'Neill B., *The Fundamental Equations of Submersions*, Michigan Math. ... (1966) 459–469.

[14] Reinhart B., *Foliated Manifolds with Bundle-Like Metrics*, Ann. Math. ... 119–132.

[15] Tondeur Ph., *Foliations on Riemannian Manifolds*, Springer, New York ...