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TOPOLOGY OF SINGULAR FOLIATIONS

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We prove that the set of leaves of a singular foliation with the Nishimori relation is partially ordered if and only if all leaves are proper. Bibliography: 9 *titles.*

1 Introduction

1.1. Let M be a smooth n-dimensional manifold, and let A be a maximal atlas defining the structure of smooth manifolds of class C^r on M, where $r \geqslant 0$. The manifold M is a manifold of class C^s if $0 \leqslant s \leqslant r$. A system of local curvilinear coordinates on the C^s-manifold M is denoted by A^s. Let $0 < k < n$.

Definition 1.1 ([1])**.** A family $F = \{L_\alpha; \alpha \in B\}$ of linearly connected subsets of M is called a *k-dimensional C^s-foliation* if it satisfies the following conditions:

(F_{I}) $\bigcup_{\alpha\in B} L_\alpha = M$,

(F_{II}) for all $\alpha, \beta \in B$, if $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \varnothing$,

(F_{III}) for any point $p \in M$ it is possible to choose local coordinates $(U_\lambda, \varphi_\lambda) \in A^{(s)}$, $p \in U_\lambda$ such that if $U_\lambda \cap L_\alpha \neq \varnothing$ for some $\alpha \in B$, then the components of the linear connectedness of the set $\varphi_\lambda(U_\lambda \cap L_\alpha)$ have the form

$$\{(x_1, x_2, \dots, x_n) \in \varphi_\lambda(U_\lambda) : x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \dots, x_n = c_n\},$$

where $c_{k+1}, c_{k+2}, \dots, c_n$ are constant on the components of the linear connection.

The set L_α is called a *leaf* of the foliation F.

The presence of a foliation F on a manifold M is written as (M, F).

Assumptions (F_{I}) and (F_{II}) mean that M consists of mutually disjoint leaves, whereas (F_{III}) means that leaves are locally arranged as parallel planes. If Assumption (F_{III}) holds, then the coordinate neighborhoods $(U_\lambda, \varphi_\lambda) \in A^{(s)}$ are said to be *foliated* and the collection of all

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foliated coordinate neighborhoods is denoted by A_F^s and called the *system of foliated coordinate neighborhoods.*

Example 1.1. The simplest example of a k-dimensional foliation is the representation of the Euclidean space R^n as the union of k-dimensional parallel planes.

Example 1.2. Let X be a vector field on M without singular points, i.e., $X(x) \neq 0$ for all $x \in M$. By the theorem on the existence and uniqueness of a solution to the differential equation

$$\dot{x} = X(x), \tag{1.1}$$

for each point $x_0 \in M$ only one integral curve $\gamma(t, x_0)$ of the vector field X passes through X_0. Therefore, the manifold M is the union of integral curves of the vector field X.

The partition of M into integral curves of the vector field X is a one-dimensional foliation. Assumption ($\mathrm{F_{III}}$) holds by the rectification theorem [2] asserting that for each point $x_0 \in X$ such that $X(x_0) \neq 0$ there exists a neighborhood U and a local curvilinear coordinate system $(y_1, y_2, \ldots, y_n)$ on U such that, in these coordinates, Equation (1.1) takes the form

$$\begin{aligned} \dot{y}_1 &= 1, \\ \dot{y}_2 &= 0, \\ &\cdots\cdots \\ \dot{y}_n &= 0. \end{aligned}$$

Let $f : M \to N$ be a differentiable mapping of maximum rank, where M is a smooth n-dimensional manifold and N is a smooth m-dimensional manifold. The mapping f is called a *submersion* if $n > m$ and *immersion* if $n < m$. The following theorem asserts that differentiable submersions generate smooth foliations.

Theorem 1.1 ([1])**.** *Let $f : M \to N$ be a submersion. Then for each point $q \in N$ the set $L_q = \{p \in M : f(q) = q\}$ is an $(n - m)$-dimensional manifold and the partition of M into manifolds L_q is a $k = (n - m)$-dimensional foliation.*

Let (M, F) be a foliated manifold, and let L_α be a leaf of the foliation F. The system of foliated coordinate neighborhoods A_F^s defines the topology A_α on every leaf L_α.

Let $l_\alpha : (L_\alpha, A_\alpha) \to M^n$ be the inclusion map from (L_α, A_α) to M^n. By Assumption ($\mathrm{F_{III}}$), the one-to-one mapping l_α is a C^r–immersion.

If l_α is a C^r–embedding, we say that L_α is a *proper leaf.* It is easy to see that each compact leaf is proper. If Int $\overline{L}_\beta \neq \oslash$, then the leaf l_β is said to be *locally dense.* A leaf that is neither proper nor locally dense is called *exceptional.*

1.2. We consider completely integrable distributions and related singular foliations.

Definition 1.2 ([3])**.** Let M be a smooth n-dimensional manifold, and let T_xM be the tangent space at a point $x \in M$. A mapping P sending each point $x \in M$ to some subspace $P(x) \subset T_xM$ is called a *distribution.* A distribution P is *k–dimensional* if dim $P(x) = k$ for all $x \in M$.

Definition 1.3 ([4])**.** A distribution P is *smooth* if for each point $x \in M$ there exists a neighborhood $U(x)$ of x and smooth vector fields X_1, X_2, , X_m defined on $U(x)$ and such that the vectors $X_1(y)$, $X_2(y)$, , $X_m(y)$ form a basis for the subspace $P(y)$ for each $y \in U(x)$.

Definition 1.4. A distribution P is *completely integrable* if for each point $x \in M$ there exists a connected submanifold N_x of M containing x and such that $T_y N_x = P(y)$ for all $y \in N_x$. The submanifold N_x is called an *integral manifold* of the distribution P.

We say that a vector field X belongs to a distribution P if $X(x) \in P(x)$ for all $x \in M$. Recall that a distribution P of constant dimension on a manifold M is called *involutive* if $X, Y \in P$ implies $[X, Y] \in P$.

The following assertion, known as the Frobenius theorem, provides a necessary and sufficient condition for a distribution of constant dimension to be completely integrable.

Theorem 1.2 (Frobenius theorem; see [5])**.** *A distribution P of a constant dimension is completely integrable if and only if it is involutive.*

It is clear that, if the distribution P is completely is integrable, then the manifold M decomposes into integral manifolds of this distribution. If the distribution P is completely integrable and the dimension $k = \dim P(x)$ is independent of x, then the integral manifolds generate a k-dimensional foliation. In the general case, since the dimensions of integral manifolds are distinct, no foliation arises on M. This fact leads to the notion of singular foliations.

Definition 1.5 ([6])**.** A subset L of M is called a k-dimensional leaf if there is a differential structure σ on L such that

(a) (L, σ) is a connected k-dimensional immersed manifold M,

(b) if N is an arbitrary locally connected topological space and $f : N \to M$ is a continuous mapping such that $f(N) \subset L$, then $f : N \to (L, \sigma)$ is continuous.

It follows from the definition of an immersion that, if $f : N \to M$ is a differentiable map and $f(N) \subset L$, then $f : N \to (L, \sigma)$ is also differentiable. In particular, σ is the only differential structure, with respect to which L is a k-dimensional submanifold.

Definition 1.6 ([7])**.** Let $1 \leqslant q \leqslant \infty$ or $q = \omega$ (analyticity). We say that a partition F of M into C^q-leaves is a *singular foliation* of class C^q if for each point $x \in M$

(a) there is a local C^q-map ψ with the domain $U \times V$, where U is a neighborhood of zero in R^k, V is a neighborhood of zero in R^{n-k}, and k is the dimension of the leaf passing through the point x,

(b) $\psi(0, 0) = x$,

(c) if L is a leaf of F, then $L \cap \psi(U \times V) = \psi(U \times l)$, where $l = \{v \in V : \psi(0, v) \in L\}$.

As in the case of foliations of constant dimension, the leaf of F passing through a point x is denoted by $L(x)$.

Example 1.3. Each regular foliation is singular. In this case, each connected component of l is a point.

Example 1.4. Let S be a closed subset of M, where $M \setminus S = \bigcup\limits_{\alpha} M_\alpha$ is a partition of $M \setminus S$ into connected components M_α and F_α is a regular foliation on M_α. Then the family $F = \bigcup\limits_{\alpha} F_\alpha \cup \{x : x \in S\}$ defines a foliation with singularities on M.

Example 1.5. As follows from Theorem 1.3, the orbits of a system of vector fields generate a singular foliation.

Let us recall the notion of an orbit. Let D be a finite or infinite family of smooth vector fields defined on M. For $X \in D$ we denote by $X^t(x)$ the integral curve of X passing through a point $x \in M$ at $t = 0$. The mapping $t \to X^t(x)$ is defined in some domain $I(x)$ that, in general, depends not only on the field X, but also on the initial point x. Below, we assume $t \in I(x)$.

Definition 1.7. An orbit $L(x)$ of a family D of vector fields passing through a point x is the set of points y in M such that there exist real numbers $t_1, t_2, \ldots, t_k$ and vector fields $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ in D such that $y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(\ldots(X_{i_1}^{t_1})\ldots))$, where k is an arbitrary positive integer.

It is obvious that, if a family D consists of a single vector field, then the orbit is a smooth curve. The fundamental result in the study of geometric and topological properties of orbits is given by the Sussmann theorem [4] asserting the existence of a completely integrable distribution P on M such that for each $x \in M$ the orbit $L(x)$ coincides with the maximal integral submanifold of P passing through x.

The topology of the orbit $L(x_0)$ (the Sussmann topology) is introduced as the strongest topology in which all mappings of the form $(t_1, t_2, \ldots, t_k) \longrightarrow X_1^{t_1}(X_{i_{k-1}}^{t_{k-1}} \ldots (X_{i_1}^{t_1}(x_0)\ldots))$ are continuous.

Theorem 1.3 ([6]). *Let M be a smooth (of class C^{r+1}) manifold of dimension n, and let D be a system of vector fields of class C^r, where $r \geqslant 1$. Then the orbits of the system D define a singular foliation of class C^r.*

Geometry and topology of singular foliations were studied, for example, in [5] and [7]–[8].

We note that the function $x \to \dim L(x)$ is semicontinuous from below. Indeed, if $x \in M$, $(U \times V, \psi)$ is a local chart in a neighborhood of the point x described in the definition and $L \cap \psi(U \times V) \neq \oslash$, where L is a leaf of F, then $\dim(L \cap \psi(U \times V)) = \dim(U \times l) \geqslant \dim U = \dim L(x)$. As in the case of regular foliations, a leaf L of a foliation F with singularities is said to be *proper* if the canonical injection $i : L \to M$ is an embedding.

We recall some properties of leaves of a foliation F. Let L be a leaf of a foliation F, and let x_1 and x_2 be points on L. The following lemmas are simple consequences of the theorem on continuous dependence of a solution to a differential equation on the initial data.

Lemma 1.1. *Let L be a leaf of a foliation F, and let x_1 and x_2 be points on L. Then for each neighborhood V of x_2 (in the topology of M) there exists a neighborhood U of x_1 such that $V \cap L_\alpha \neq \oslash$, where L_α is a leaf of F and $U \cap L_{\alpha \neq \oslash}$.*

Lemma 1.2. *Let U be an open set in M. Then the union of the leaves intersecting U is an open set.*

Lemma 1.3. *A leaf L of a foliation F is improper if and only if $\overline{L \backslash K} = \overline{L}$ (the closure in M) for every compact set K in L.*

Proof. *Sufficiency* is obvious.

Necessity. Assume that K is a compact set in L, $x \in K$, $V(x)$ is a neighborhood of x in the topology of the layer L, and $V_k(x)$ is the fundamental system of neighborhoods of x in the topology of M. Since L is an improper leaf, $V(x)$ is not a neighborhood in the topology induced from M, i.e., for each k the set $V_k(x) \cap (L \backslash V_{(x)})$ is not empty.

Let $x_k \in V_k(x) \cap (L \backslash V(x))$. It is obvious that the sequence x_k converges to x in the topology of M, but this is not the case in the topology of L. Thus, $\overline{L \backslash V(x)} = \overline{L}$. Since the set K can be covered by finitely many neighborhoods, we conclude that $\overline{L \backslash K} = \overline{L}$. □

2 The Main Results

We denote by M/F the set of leaves of a foliation F and introduce a relation $\leqslant$ on the set M/F. Following [9], we introduce the notion of depth for leaves and foliations.

Let $L^1, L^2 \in M/F$. We write $L^1 \leqslant L^2$ if $L^1 \subset \overline{L^2}$ or $L^1 < L^2$ if $L^1 \leqslant L^2$ and $L^1 \neq L^2$. Denote by $(M/F, \leqslant)$ the set of leaves with the relation $\leqslant$. It is obvious that the relation $\leqslant$ is reflexive, but, in many cases, is not antisymmetric (for example, such a situation occurs for the irrational winding of a torus). Thus, in the general case, the set $(M/F, \leqslant)$ is not partially ordered.

The depths of a leaf L and a foliation F are defined by

$$\mathrm{d}\, L = \mathrm{Sup}\{k : \text{there are } k \text{ leaves such that} L^0 < L^1 < L^2 < \quad < L^k = L\},$$

$$\mathrm{d}\, F = \mathrm{Sup}(\mathrm{d}\, L : L \in M/F).$$

For foliations of codimension one on a compact manifold the following result was proved in [9].

Theorem 2.1 ([9])**.** *If* $\mathrm{d}\, F < \infty$ *or all leaves of a foliation* F *are proper, then the set* $(M/F, \leqslant)$ *is partially ordered.*

Some questions concerning properties of foliations of codimension one were put in [9]. These questions are also of interest for more general foliations, and we formulate them below.

Question 2.1. Can a foliation F have improper leaves if the set $(M/F, \leqslant)$ is partially ordered?

Question 2.2. Can we assert that a leaf L is proper if $\mathrm{d}\, L < \infty$?

Question 2.3. Does there exist a foliation F such that the set $(M/F, \leqslant)$ is partially ordered and $\mathrm{d}\, F = \infty$?

The following theorem answers Questions 2.1 and 2.2.

Theorem 2.2. *A set* $(M/F, \leqslant)$ *is partially ordered if and only if all leaves of the foliation* F *are proper.*

Proof. *Necessity.* We note that, if a leaf is a closed subset, then it is proper [5]. Let $(M/F, \leqslant)$ be partially ordered.

We first show that each leaf is a set of type G_δ in its closure and then prove that an improper leaf L is not a set of type G_δ in its closure. Assume that a leaf $L \in M/F$ is not a closed set. Consider a point $x \in L$ and the fundamental system $V_m(x)$ of neighborhoods of x in the topology of M. Introduce the sets $A_m = \cup L_\alpha$, $L_\alpha \cap V_m(x) \neq \varnothing$, where L_α is a leaf of the foliation F. Now, it is easy to check that for each m A_m is an open set.

Assume that $A = \bigcap A'_m$ and $A'_m = A_m \bigcap N$, where $N = \overline{L}$ is the closure of a leaf L in the topology of M. Let us show that $A = L$. Indeed, it is clear that $L \subset A$. Let $y \in A$. Since the intersection of invariant sets is invariant, $L(y) \subset A$. Therefore, the set $L(y) \cap V_m(x)$ is nonempty for each m and $L(y) \subset N$. Since $x \in \overline{L}(y)$ and $L(y)$ is an invariant set, we have $L(x) \subset \overline{L}(y)$. Thus, $L(y) \leqslant L$ and $L \leqslant L(y)$. By the assumptions of the theorem, $L = L(y)$. Hence $A = L$ and the leaf L is a set of type G_δ in $N = \overline{L}$.

Let us show that an improper leaf L is not a set of type G_δ in $N = \overline{L}$. Each leaf L of F can be represented as $L = \bigcup\limits_{i=1}^{\infty} K_i$, where K_i are compact sets in L. By Lemma 1.3, an improper leaf

L is a set of the first category in N. The set N, regarded as a closed subset of a locally compact space M, is a locally compact space and, consequently, possesses the Baire property. As known, if a leaf L is a set of type G_δ in N, then L is nowhere dense in N. A contradiction obtained proves the required assertion.

Sufficiency. Assume that all leaves of F are proper and the set $(M/F, \leqslant)$ is not partially ordered. Then there are two different leaves L_1 and L_2 such that $L_1 \subset L_2$ and $L_2 \subset L_1$. Let $x \in L_1$ and let $V'(x)$ be a neighborhood of x in the topology of L_1. Since the leaf L_1 is proper, we can assume that there exists a neighborhood $V(x)$ of x in the topology of M such that $V'(x) = V(x) \cap L_1$. Since $L_1 \subset L_2$, there is a point $y \in V(x) \cap L_2$ and a neighborhood $V(y)$ in the topology of M such that $V(y) \subset V(x)$ and $V(y) \bigcap V\prime(x)) = \varnothing$. Since $L_2 \subset L_1$, we have $V(y) \cap L_1 \neq \varnothing$ and, consequently, $V(x) \cap L_1 \neq V'(x)$. A contradiction obtained proves the theorem. □

The following assertions follow from the proof of Theorem 2.2.

Corollary 2.1. *The closure of each improper leaf L contains an uncountable number of improper leaves such that their closure coincides with L.*

Indeed, it follows from the proof of Theorem 2.2 that the closure of each leaf L_α of A coincides with $\overline{L}$ and $A \neq L$. Hence A contains other leaves different from L_α. If there are at most countably many such leaves, then $L_\alpha = A \backslash (\bigcup L_i)$, i.e., the union of a countable number of leaves $\bigcup L_i$ is a set of type F_δ (since each leaf is of type F_δ) and the set A is a set of type G_δ, which implies that L_α is a set of type G_δ.

By Corollary 2.1, if a foliation with singularities has an improper leaf, then it has an uncountably many improper leaves. Note that the number of proper leaves can be finite or countable.

Corollary 2.2. *If* $\mathrm{d}\, L < \infty$, *then L is a proper leaf.*

Hence, if L is an improper leaf, then Corollary 2.2 implies $\mathrm{d}\, L = \infty$.

For foliations of codimension one on compact manifolds this assertion was proved in [9].

In general, the depth of a proper leaf can be infinite. There are examples of foliations of codimension one such that the closure of a proper leaf contains improper leaves. It is easy to construct examples of singular foliations possessing proper leaves of infinite depth.

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The author declares that there is no conflict of interest.

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