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Discrete sine-Gordon equation on metric graphs: A simple model for Josephson junction networks

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Abstract

We consider discrete sine-Gordon equation on branched domains. The latter is modeled in terms of the metric graphs with discrete bonds having the form of the branched 1D chains. Exact analytical solutions of the problem are obtained for special case of the constraints given in terms of a simple sum rule. Numerical solution is obtained when the constraint is not fulfilled. A simple model of a Josephson junction network is proposed using the obtained results.

1. Introduction

Sine-Gordon solitons attracted much attention in different contexts, from solid state mechanics to quantum field theory and Josephson junctions (see, [1–9]). Recently, special attention was given to the study of sine-Gordon solitons in branched domains and networks. Particle and wave dynamics in low-dimensional branched domains is of importance for many practically significant problems arising in newly emerging technologies. As many device structures and functional materials have branched or networked form, controlling of wave propagation in such structures play crucial role for device optimization and material design. Solving such a task requires developing realistic and physically acceptable models of the wave dynamics in low-dimensional branched structures. An effective way for modelling of solitons in networks can be based on the solution of different nonlinear evolution equations (approving soliton solutions) on metric graphs. This kind of problem has become subject of extensive study recently (see, [10–26]). Especially, this concerns condensed matter systems, where particle and wave transport is in linear(quantum) and nonlinear regimes, when the problem of tunable transport is of crucial importance. Such tasks appear, e.g., in BEC dynamics, optical harmonic generation in low-dimensional quantum materials, Josephson junctions, etc. Tuning the wave propagation process in these structures allows optimization of material's functional properties and improving of the device performance. In this paper we consider the model, which is directly related to the problem of soliton dynamics in Josephson junction networks. Namely, we consider the sine-Gordon equation on a discrete branched lattice. The sine-Gordon equation in such lattice can describe Josephson junction (JJ) network consisting of branched JJ-arrays, where superconducting leads are separated by point-like insulators or normal metals. Different versions of such branched JJ-arrays have been considered earlier in the [27–32]. However, these studies did not consider soliton dynamics in such structures and did not use metric graph based approach for the sine-Gordon equation. In such lattice, the phase difference (on each junction) between the leads is described in terms of the discrete sine-Gordon equation. Imposing the boundary conditions in the form of weight continuity and Kirchhoff rules at the branching point, we derive constraints ensuring integrability of the discrete sine-Gordon equation in metric graph. Such constraint can be written in the form of a simple sum rule in terms of the nonlinearity coefficients. Apart from the branched Josephson junction arrays, within the Frenkel-Kontorova model [2, 3], the discrete sine-Gordon equation on metric graphs can be used for modeling deformation propagation in branched solid materials. In both cases, the main problem having practical interest is tunable

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propagation of sine-Gordon soliton in a branched structure. Here we show that in case, when the problem is integrable, sine-Gordon solitons pass through the graph vertex without reflection, i.e. there is no backscattering at the branching points for integrable case.

The paper is organized as follows. In the next section we briefly recall the discrete sine-Gordon equation on a line. In section III we present formulation of the task and its solution for star branched graph. Section IV extends the study for the case of loop graph. In section V a model of Josephson network is proposed. Finally, the section VI provides some concluding remarks.

2. Discrete sine-Gordon equation on a line

The discrete sine-Gordon (DSG) equation follows from the Hamiltonian of the Frenkel-Kontorova model, which is given as [2]

$$H = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2}\left(\frac{du_n}{dt}\right)^2 + \beta(1-\cos u_n) + \frac{a}{2}(u_{n+1}-u_n)^2\right].$$

equation of motion for this Hamiltonian, leads to the following standard discretized sine-Gordon equation:

$$\frac{d^2u_n}{dt^2} - a(u_{n+1} - 2u_n + u_{n-1}) + \beta\sin u_n = 0, \tag{2}$$

where a and β are constant coefficients. For the above DSG equation, the formula for calculating the charge can be written as [3, 7]

$$Q = \frac{1}{2\pi}\sqrt{\frac{a}{\beta}}\sum_{n=-\infty}^{+\infty}(u_{n+1}-u_n). \tag{3}$$

The other conservative quantities are the energy given by equation (1) and momentum, which is given by

$$P = \frac{\sqrt{a}}{\beta}\sum_{n=-\infty}^{+\infty}\frac{du_n}{dt}(u_{n+1}-u_n). \tag{4}$$

Kink and antikink soliton solution of equation (2) can be written as (for $a = 1$ and $\beta = 1$)

$$u_n(t) = 4\arctan\left[\exp\left(\pm\frac{(n-n_0)-vt}{\sqrt{1-v^2}}\right)\right], \tag{5}$$

where v is the velocity of the kink (or antikink) soliton. Besides kink, equation (2) has also breather solution, which is given as [7] (for $a = 1$ and $\beta = 1$)

$$u_n(t) = 4\arctan\left[\frac{\sqrt{1-\omega^2}\sin(\omega t)}{\omega\cosh(\sqrt{1-\omega^2}(n-n_0))}\right], \tag{6}$$

where ω is real parameter of breather solution.

In the next section we use this solution to construct soliton solution of DSG equation on a graph.

3. Discrete sine-Gordon equation on a star graph

In this section we formulate the problem for the star shaped graph. The study is limited to the three bond star graph. We note that the extension to the star graph with arbitrary number of bonds is straightforward.

Consider the star graph that consists of three semi-infinite chains connected at the vertex (see figure 1). For the first bond ($j = 1$) n is numbered as $n \in b_1 = \{-1, -2, -3, \ldots\}$, where $(j, -1)$ stands for the point nearest to the vertex. For the right handed bonds ($j = 2, 3$) n is numbered as $n \in b_j = \{0, 1, 2, \ldots\}$, where $(j, 0)$ means the branching point.

The DSG equation is written on the each bond of the star graph as follows

$$\frac{d^2u_{j,n}}{dt^2} - a_j(u_{j,n+1} - 2u_{j,n} + u_{j,n-1}) + \beta_j\sin u_{j,n} = 0. \tag{7}$$

The charge for equation (7) can be written as

$$Q = \frac{1}{2\pi}\sum_{j=1}^{3}\sqrt{\frac{a_j}{\beta_j}}\sum_{b_j}(u_{j,n+1}-u_{j,n}). \tag{8}$$

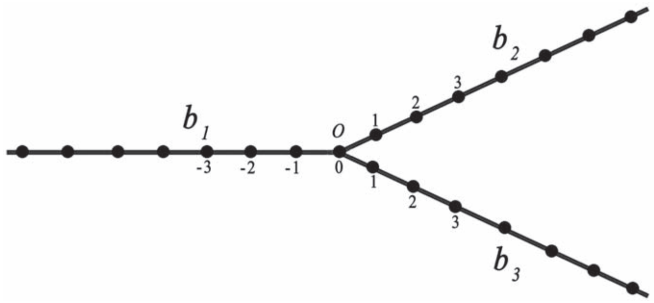

Figure 1. Star branched 1D lattice.

At the sites, $(j, n) = \{(1, 0), (2, -1), (3, -1)\}$, following the Refs. [11, 12], we assumed the below relations:

$$\begin{aligned} u_{1,0} &= \alpha_2^{(1)} u_{2,0} + \alpha_3^{(1)} u_{3,0}, \\ u_{2,-1} &= \alpha_1^{(2)} u_{1,-1} + \alpha_3^{(2)} u_{3,0}, \\ u_{3,-1} &= \alpha_1^{(3)} u_{1,-1} + \alpha_2^{(3)} u_{2,0}, \end{aligned}$$

($\alpha_m^{(j)}$ ($j, m = 1, 2, 3$ and $j \neq m$) are constant coefficients, which will be determined later), from the charge conservation law which is given as

$$\frac{dQ}{dt} = \frac{1}{2\pi}\left[\sqrt{\frac{a_1}{\beta_1}}\frac{du_{1,0}}{dt} - \sqrt{\frac{a_2}{\beta_2}}\frac{du_{2,0}}{dt} - \sqrt{\frac{a_3}{\beta_3}}\frac{du_{3,0}}{dt}\right] = 0, \tag{9}$$

we have

$$\sqrt{\frac{a_1}{\beta_1}}\frac{du_{1,0}}{dt} = \sqrt{\frac{a_2}{\beta_2}}\frac{du_{2,0}}{dt} + \sqrt{\frac{a_3}{\beta_3}}\frac{du_{3,0}}{dt}. \tag{10}$$

Similarly, for the energy, which is given by

$$E = \sum_{j=1}^{3}\frac{1}{\beta_j}\sum_{b_j}\left[\frac{1}{2}\left(\frac{du_{j,n}}{dt}\right)^2 + \beta_j(1 - \cos u_{j,n}) + \frac{a_j}{2}(u_{j,n+1} - u_{j,n})^2\right], \tag{11}$$

we have the following conservation law:

$$\begin{aligned} \frac{dE}{dt} &= \frac{a_1}{\beta_1}\frac{du_{1,0}}{dt}(u_{1,0} - u_{1,-1}) - \frac{a_2}{\beta_2}\frac{du_{2,0}}{dt}(u_{2,0} - u_{2,-1}) \\ &- \frac{a_3}{\beta_3}\frac{du_{3,0}}{dt}(u_{3,0} - u_{3,-1}) = 0, \end{aligned}$$

that leads to the following condition:

$$\begin{aligned} \frac{a_1}{\beta_1}\frac{du_{1,0}}{dt}(u_{1,0} - u_{1,-1}) &= \frac{a_2}{\beta_2}\frac{du_{2,0}}{dt}(u_{2,0} - u_{2,-1}) \\ &+ \frac{a_3}{\beta_3}\frac{du_{3,0}}{dt}(u_{3,0} - u_{3,-1}). \end{aligned} \tag{12}$$

From equations (10) and (12) we can obtain the following 'quasi' vertex boundary conditions:

$$\sqrt{\frac{a_1}{\beta_1}}u_{1,0} = \sqrt{\frac{a_2}{\beta_2}}u_{2,0} + \sqrt{\frac{a_3}{\beta_3}}u_{3,0}, \tag{13}$$

$$\begin{aligned} \sqrt{\frac{a_1}{\beta_1}}(u_{1,0} - u_{1,-1}) &= \sqrt{\frac{a_2}{\beta_2}}(u_{2,0} - u_{2,-1}) \\ &= \sqrt{\frac{a_3}{\beta_3}}(u_{3,0} - u_{3,-1}). \end{aligned} \tag{14}$$

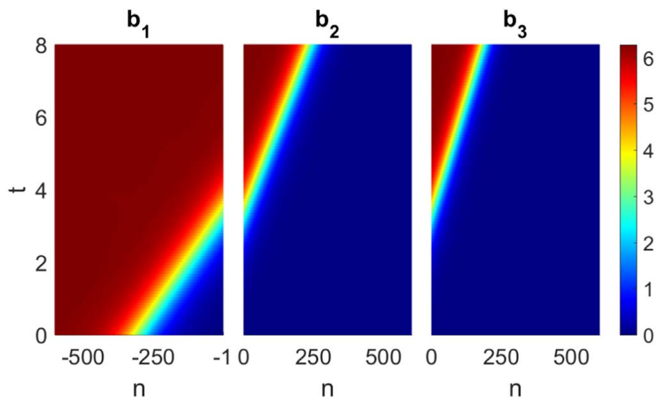

Figure 2. Contour plot of the numerical solution of DSG equation on the star graph when the sum rule (17) is fulfilled, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1, a_1 = 2.56, a_2 = 0.81, a_3 = 0.49$.

The parameters $\alpha_m^{(j)}$ introduced above can be determined here in terms of system parameters as

$$\alpha_2^{(1)} = \sqrt{\frac{a_2\beta_1}{a_1\beta_2}},\quad \alpha_3^{(1)} = \sqrt{\frac{a_3\beta_1}{a_1\beta_3}},$$
$$\alpha_1^{(2)} = \sqrt{\frac{a_1\beta_2}{a_2\beta_1}},\quad \alpha_3^{(2)} = -\sqrt{\frac{a_3\beta_2}{a_2\beta_3}},$$
$$\alpha_1^{(3)} = \sqrt{\frac{a_1\beta_3}{a_3\beta_1}},\quad \alpha_2^{(3)} = -\sqrt{\frac{a_2\beta_3}{a_3\beta_2}}.$$

From equations (13) and (14) one can define $u_{j,n}$ at the virtual $(j, n) = \{(1, 0), (2, -1), (3, -1)\}$ sites:

$$u_{1,0} = \sqrt{\frac{\beta_1}{a_1}}\left(\sqrt{\frac{a_2}{\beta_2}}u_{2,0} + \sqrt{\frac{a_3}{\beta_3}}u_{3,0}\right),$$
$$u_{2,-1} = \sqrt{\frac{\beta_2}{a_2}}\left(\sqrt{\frac{a_1}{\beta_1}}u_{1,-1} - \sqrt{\frac{a_3}{\beta_3}}u_{3,0}\right),$$
$$u_{3,-1} = \sqrt{\frac{\beta_3}{a_3}}\left(\sqrt{\frac{a_1}{\beta_1}}u_{1,-1} - \sqrt{\frac{a_2}{\beta_2}}u_{2,0}\right). \tag{15}$$

Then the soliton (kink) solution of equation (7) can be written in terms of solution of equation (2) as

$$u_{j,n}(t) = 4\arctan\left[\exp\left(\pm\frac{\sqrt{\frac{\beta_j}{a_j}}(n - n_0) - \beta_j vt}{\sqrt{1 - v^2}}\right)\right]. \tag{16}$$

Substituting the solution in equation (16) to the quasi vertex boundary conditions (equations (13) and (14)) leads to the following relations:

$$\sqrt{a_1} = \sqrt{a_2} + \sqrt{a_3},\quad \beta_1 = \beta_2 = \beta_3. \tag{17}$$

As in the previous section, the breather solution of discrete sine-Gordon equation for the star graph :

$$u_{j,n}(t) = 4\arctan\left[\frac{\sqrt{1 - \omega^2}\sin(\omega\beta_j t)}{\omega\cosh\left(\sqrt{1 - \omega^2}\sqrt{\frac{\beta_j}{a_j}}(n - n_0)\right)}\right]. \tag{18}$$

It should be noted that equation (17) presents condition (constraint) for integrability of the problem given by equations (7), (13) and (14). In other words, the discrete sine-Gordon equation (7) on metric star graph presented in figure 1 is integrable if and only if the sum rule in equation (17) is fulfilled. In figure 2, spatio-temporal evolution of the sine-Gordon soliton obtained using the exact solution given by equation (16) is plotted. One can clearly observe from this plot that transmission of the sine-Gordon soliton through the branching point of the graph is reflectionless, i.e., no backscattering is observed for the case, when the problem is integrable. This feature can be confirmed by direct computing reflection coefficient (as the ratio of soliton energy

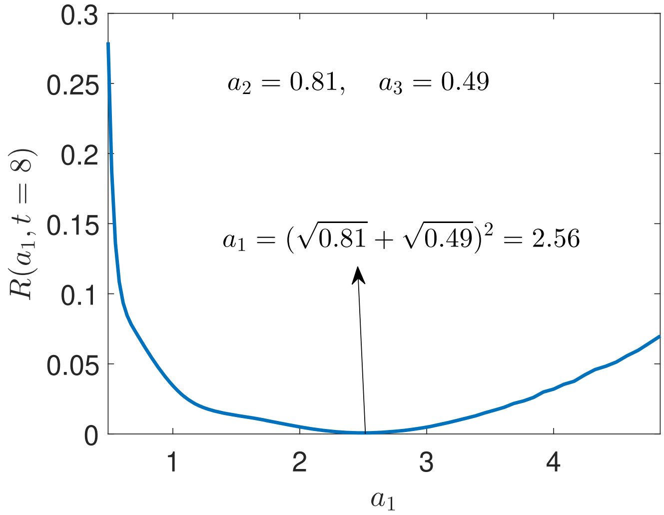

Figure 3. The reflection coefficient (determined as $R = E_1/(E_1 + E_2 + E_3)$) calculated at $t = 8$ as a function of a_1 for fixed values of a_2 and a_3.

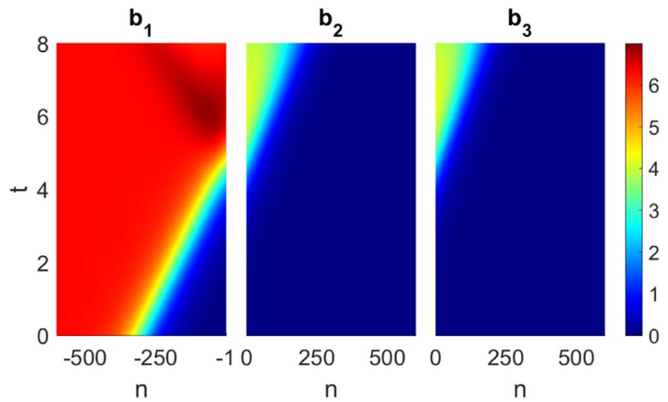

Figure 4. Contour plot of the numerical solution of DSG equation on the star graph when the sum rule (17) is broken, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1, a_1 = 1.44, a_2 = 1, a_3 = 0.81$.

on bond 1 to the total one). In figure 3 dependence of reflection coefficient, $R = E_1/(E_1 + E_2 + E_3)$ (at $t = 8$) on a_1, for the fixed $a_2 = 0.81, a_3 = 0.49$, is plotted. As it can be seen from this plot, at the value of $a_1 = 2.56$, which corresponds to fulfilling of the sum rule, R becomes zero. For the case, when the problem is not integrable, i.e. when the sum rule in equation (17) is broken, the problem need to be solved numerically. The plot of the solution for the case, when the sum rule is broken is presented in figure 4. The plot shows that transmission of sine-Gordon soliton through the branching point is accompanies by reflection.

Fig 5 presents breather solutions of equation (7) on graph obtained numerically by choosing the initial condition as a breather in equation (18) and assuming that the sum rule in equation (17) is fulfilled. The parameter ω is chosen as $\omega = 0.5$.

4. Extending to the loop graph

The above treatment of the DSG equation on star graph can be extended to the case of other graph topologies, e.g., to loop graph presented in figure 6. The graph consists of two semi infinite and two finite chains connected to each other at two vertices. On each bond of the loop graph we have the DSG equation given by (7). The charge and the energy for such structure are given as (respectively)

$$Q = \frac{1}{2\pi} \sum_{j=1}^{4} \sqrt{\frac{a_j}{\beta_j}} \sum_{b_j} (u_{j,n+1} - u_{j,n}), \quad (19)$$

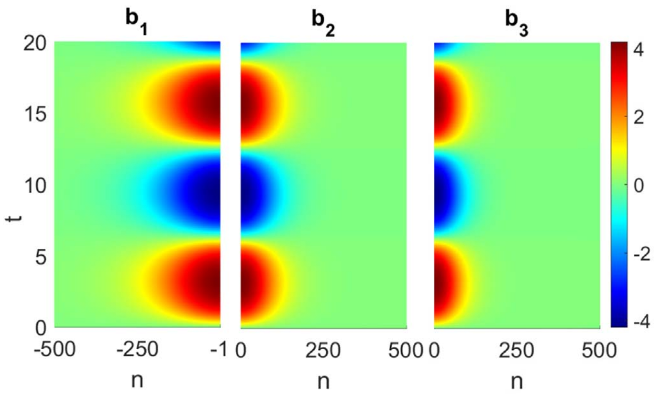

Figure 5. Contour plot of numerical solution for breather initial condition when the sum rule is fulfilled.

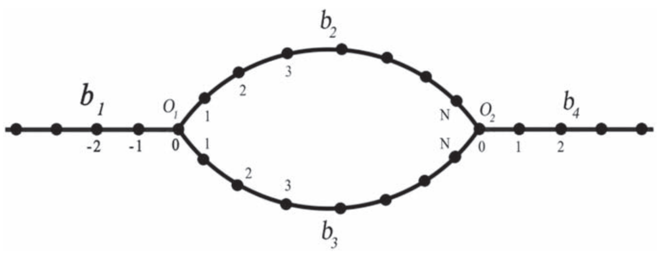

Figure 6. Discrete loop graph.

$$E = \sum_{j=1}^{4} \frac{1}{\beta_j} \sum_{b_j} \left[\frac{1}{2} \left(\frac{du_{j,n}}{dt} \right)^2 + \beta_j (1 - \cos u_{j,n}) + \frac{a_j}{2} (u_{j,n+1} - u_{j,n})^2 \right]. \tag{20}$$

From the conservation laws for these quantities we obtain the following relations:

$$\sqrt{\frac{a_1}{\beta_1}} \frac{du_{1,0}}{dt} + \sqrt{\frac{a_2}{\beta_2}} \frac{du_{2,N+1}}{dt} + \sqrt{\frac{a_3}{\beta_3}} \frac{du_{3,N+1}}{dt} = \sqrt{\frac{a_2}{\beta_2}} \frac{du_{2,0}}{dt} + \sqrt{\frac{a_3}{\beta_3}} \frac{du_{3,0}}{dt} + \sqrt{\frac{a_4}{\beta_4}} \frac{du_{4,0}}{dt}. \tag{21}$$

$$\frac{a_1}{\beta_1} \frac{du_{1,0}}{dt} (u_{1,0} - u_{1,-1}) + \frac{a_2}{\beta_2} \frac{du_{2,N+1}}{dt} (u_{2,N+1} - u_{2,N}) + \frac{a_3}{\beta_3} \frac{du_{3,N+1}}{dt} (u_{3,N+1} - u_{3,N}) = \frac{a_2}{\beta_2} \frac{du_{2,0}}{dt} (u_{2,0} - u_{1,-1}) + \frac{a_3}{\beta_3} \frac{du_{3,0}}{dt} (u_{3,0} - u_{2,-1}) + \frac{a_4}{\beta_4} \frac{du_{4,0}}{dt} (u_{4,0} - u_{3,-1}). \tag{22}$$

From equations (21) and (22) one can obtain vertex quasi-boundary conditions, which can be written as

$$\sqrt{\frac{a_1}{\beta_1}} u_{1,0} = \sqrt{\frac{a_2}{\beta_2}} u_{2,0} + \sqrt{\frac{a_3}{\beta_3}} u_{3,0},$$
$$\sqrt{\frac{a_2}{\beta_2}} u_{2,N+1} + \sqrt{\frac{a_3}{\beta_3}} u_{3,N+1} = \sqrt{\frac{a_4}{\beta_4}} u_{4,0}, \tag{23}$$

$$\begin{aligned}\sqrt{\frac{a_1}{\beta_1}}(u_{1,0} - u_{1,-1}) &= \sqrt{\frac{a_2}{\beta_2}}(u_{2,0} - u_{1,-1})\\ &= \sqrt{\frac{a_3}{\beta_3}}(u_{3,0} - u_{2,-1})\\ \sqrt{\frac{a_2}{\beta_2}}(u_{2,N+1} - u_{2,N}) &= \sqrt{\frac{a_3}{\beta_3}}(u_{3,N+1} - u_{3,N})\\ &= \sqrt{\frac{a_4}{\beta_4}}(u_{4,0} - u_{3,-1}).\end{aligned} \tag{24}$$

From equations (23) and (24) one can define $u_{j,n}$ at the virtual $(j, n) = \{(1, 0), (2, -1), (3, -1)\}$ and $(j, n) = \{(2, N+1), (3, N+1), (4, -1)\}$ sites

$$\begin{aligned}u_{1,0} &= \sqrt{\frac{\beta_1}{a_1}}\left(\sqrt{\frac{a_2}{\beta_2}}u_{2,0} + \sqrt{\frac{a_3}{\beta_3}}u_{3,0}\right),\\ u_{2,-1} &= \sqrt{\frac{\beta_2}{a_2}}\left(\sqrt{\frac{a_1}{\beta_1}}u_{1,-1} - \sqrt{\frac{a_3}{\beta_3}}u_{3,0}\right),\\ u_{3,-1} &= \sqrt{\frac{\beta_3}{a_3}}\left(\sqrt{\frac{a_1}{\beta_1}}u_{1,-1} - \sqrt{\frac{a_2}{\beta_2}}u_{2,0}\right),\end{aligned} \tag{25}$$

and

$$\begin{aligned}u_{2,N+1} &= \sqrt{\frac{\beta_2}{a_2}}\left(\sqrt{\frac{a_4}{\beta_4}}u_{4,0} - \sqrt{\frac{a_3}{\beta_3}}u_{3,N}\right),\\ u_{3,N+1} &= \sqrt{\frac{\beta_3}{a_3}}\left(\sqrt{\frac{a_4}{\beta_4}}u_{4,0} - \sqrt{\frac{a_2}{\beta_2}}u_{2,N}\right),\\ u_{4,-1} &= \sqrt{\frac{\beta_4}{a_4}}\left(\sqrt{\frac{a_2}{\beta_2}}u_{2,N} + \sqrt{\frac{a_3}{\beta_3}}u_{3,N}\right).\end{aligned} \tag{26}$$

Again, as it was done in the case of star branched graph, by substituting the solution of DSGE on a line into the vertex quasi-boundary conditions given by equations (23) and (24), one can obtain the following constraints:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4,\ \sqrt{a_1} = \sqrt{a_2} + \sqrt{a_3},\ a_4 = a_1. \tag{27}$$

Fulfilling of this constraint implies that the problem given by equations (7), (23) and (24) is integrable and its (kink) soliton solution can be written as

$$u_{j,n}(t) = 4\arctan\left[\exp\left(\pm\frac{\sqrt{\frac{\beta_j}{a_j}}(n - n_{0j}) - \beta_j vt}{\sqrt{1 - v^2}}\right)\right], \tag{28}$$

where $n_{0j} = n_0$ for $j = 1, 2, 3$ and $n_{0j} = n_0 + N + 1$ for $j = 4$.

In figure 7 the contour plot of the soliton solution for the case, when the sum rule in equation (27) is fulfilled is presented. One can again observe absence of the backscattering in case, when the problem is integrable. Figure 8 presents similar plots for non-integrable case, when the solutions are obtained numerically. Appearing of reflection in transmission of soliton through the branching point can be clearly seen from the plot.

We can also write a breather solution of discrete sine-Gordon equation on the loop graph like the star graph as

$$u_{j,n}(t) = 4\arctan\left[\frac{\sqrt{1 - \omega^2}\sin(\omega\beta_j t)}{\omega\cosh\left(\sqrt{1 - \omega^2}\sqrt{\frac{\beta_j}{a_j}}(n - n_{0j})\right)}\right]. \tag{29}$$

5. Josephson junction network

The problem given by equations (7), (13) and (14), can be applied for modelling of different mechanical (e.g., deformation propagation in branched solid materials) and electronic systems. Here we will apply the results of section III to a simple Josephson junction network. Sketch for such a device, consisting of an arrays of Josephson

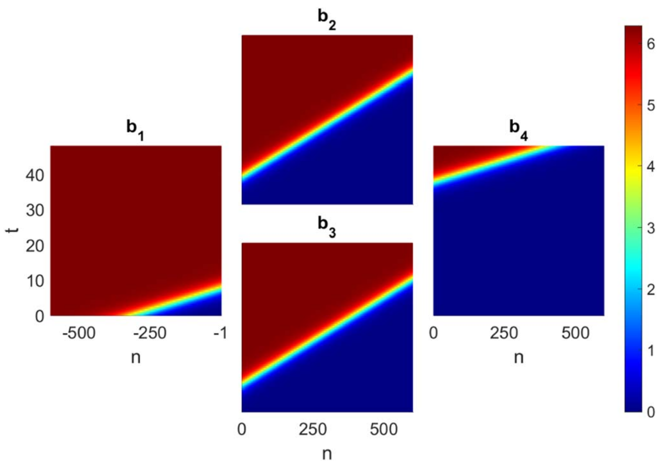

Figure 7. Contour plot of the numerical solution of DSG equation on the loop graph when the sum rule (27) is fulfilled, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1, a_1 = a_4 = 1, a_2 = a_3 = 0.25$.

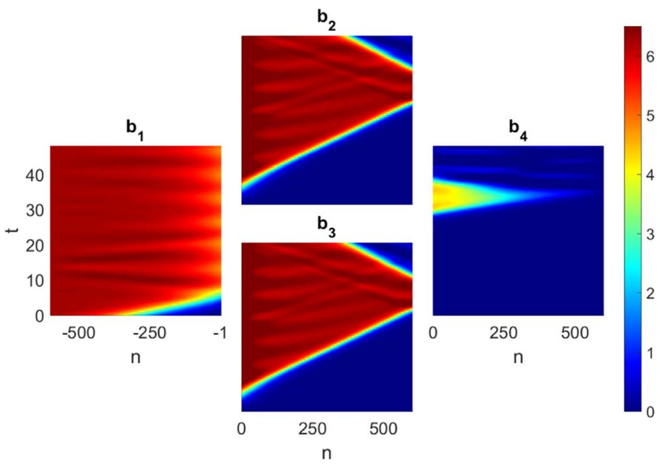

Figure 8. Contour plot of the numerical solution of DSG equation on the loop graph when the sum rule (27) is broken, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1, a_1 = a_4 = 2.25, a_2 = a_3 = 0.25$.

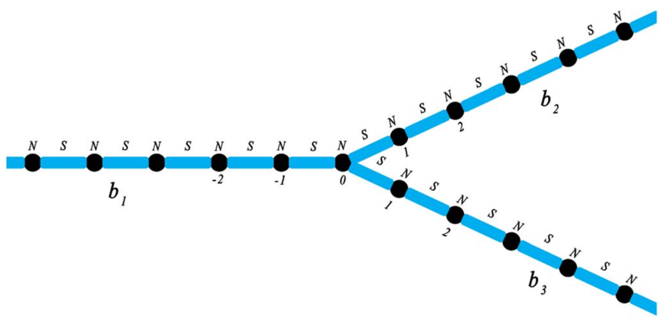

Figure 9. Branched Josephson junction.

junctions having the form of a branched lattice is presented in figure 9 for the simplest star branching. Phase difference on each site in a given branch of such structure can be described in terms of the discrete sine-Gordon equation given by equation (7) together with the boundary conditions (13) and (14). One of interesting characteristics of such device is Josephson current, which is given by

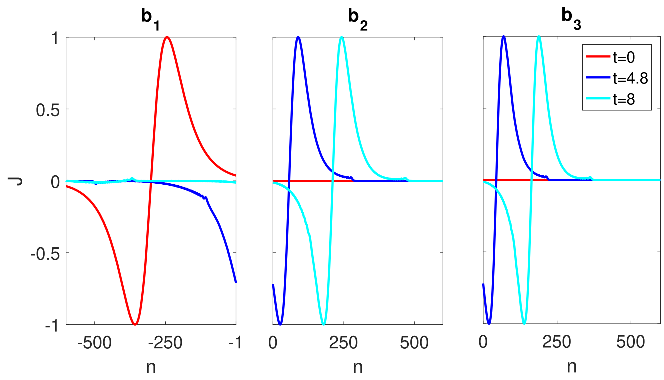

Figure 10. Josephson current on the Josephson junction network (see, figure 9), when the sum rule (17) is fulfilled, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1, a_1 = 2.56, a_2 = 0.81, a_3 = 0.49$.

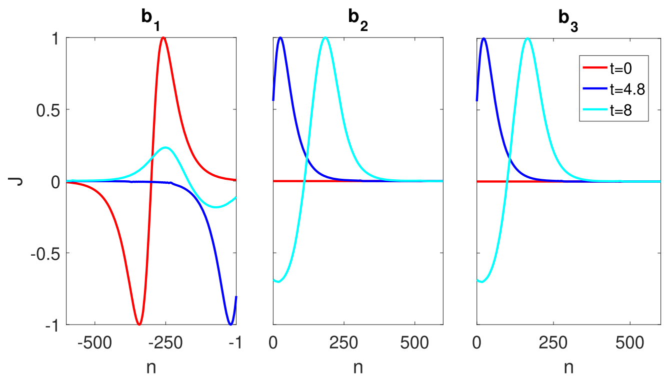

Figure 11. Josephson current on the Josephson junction network, when the sum rule (17) is broken, i.e., the parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1, a_1 = 1.44, a_2 = 1, a_3 = 0.81$.

$$J_j(n,\ t) = J_{0j} \sin[u_{j,n}(t)], \tag{30}$$

where J_{0j} is the amplitude of the Josephson current. Furthermore, we consider behavior of the current in two cases: Integrable case, when the sum rule in equation (17) is fulfilled and for the case when the sum rule is broken. Profiles of the current on each bond of the star graph at different time moments are presented in figure 10 for the case when the sum rule is fulfilled. Symmetry between the plots in second and third branches can be explained by the fact that initial condition was chosen on the first branch only, i.e. at $t = 0$ incoming soliton was fixed on the first branch. Absence of current after time elapses (at $t = 8$ in our case) can be observed from this plot, that occurs in accordance with the above statement (see, figure 2 and its description) about the reflectionless transmission of solitons in case, when the sum rule is fulfilled. Different behavior of current can be seen in figure 11, where similar plots of current are presented for the case, when sum rule given by equation (17) is broken: Current does not disappear in the first bond in this case. Profiles of the current in second and third bonds are the same due to the choice of the initial condition as incoming soliton in the first bond. Similar treatment can be done for other Josephson junction networks, having different (than the above) branching topologies, e.g., loop, tree, etc. Apparently, for more complicated branching topologies dynamics of solitons should be richer that provides more tools for tuning the electronic properties of the device.

6. Conclusions

We studied the problem of discrete sine-Gordon equation on a branched lattice by addressing the problem of integrability and soliton solutions. It is shown that the problem approves exact soliton solutions, provided the nonlinearity coupling constant (penetration depth in case of Josephson junction) assigned to each bond of the graph fulfill certain sum rule. Such case associated also with the reflectionless transmission of sine-Gordon solitons through the branching point. For the cases, when the sum rule is not fulfilled the problem solved numerically. It is shown for the latter case that reflection of soliton at the vertex can be observed. Although the above treatment dealt with the star and loop graphs, the approach developed here can be directly applied for arbitrary branching topology. In addition, one can extend the above approach to the case of networks, having alternate nonlinearity or dispersion. The obtained results are applied for modeling of the dynamics of sine-Gordon solitons in branched arrays of Josephson junctions. It is important to note that the study can be extended to the case of networks, having alternate nonlinearity or dispersion. Apart from the Josephson junction networks, the above approach can be used for modeling so-called granular superconductors, which also may form discrete and branched structure of Josephson junctions (see, e.g., [33–41]), as well as for modeling deformation propagation in branched granular solid materials (so-called granular networks). Finally, we note that the above approach can be used for the graphs of arbitrary topology (branching architecture). In case of complicated topologies, for numerical solution, one can use machine learning based numerical methods developed in the [42, 43].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Orfanidis S J 1978 *Phys. Rev. D* **18** 3822–7
[2] Braun O M and Kivshar Yu S 1998 *Phys. Rep.* 1–108 306
[3] Braun O and Kivshar Y 2013 *The Frenkel–Kontorova Model: Concepts, Methods, and Applications* (Springer)
[4] Cuevas-Maraver J and Kevrekidis P G 2014 *Floyd Williams, Nonlinear Systems and Complexity* **10**
[5] Barone A and Paternò G 1982 *Physics and Applications of the Josephson Effect* (Wiley)
[6] Likharev K K 1986 *Introduction into Dynamics of Josephson Junctions and Circuits*
[7] Cuevas-Maraver J, Kevrekidis P and Williams F 2014 *The Sine-Gordon Model and Its Applications: From Pendula and Josephson Junctions to Gravity and High-Energy Physics* (Springer International Publishing)
[8] Askerzade I, Bozbey A and Cantürk M 2018 *Modern Aspects of Josephson Dynamics and Superconductivity Electronics (Mathematical Engineering)* vol 2017 1st edn (Springer) Softcover reprint of the original
[9] Susanto H, van Gils S, Doelman A and Derks G 2005 *Phys. Lett. A* **338** 239
[10] Dutykh D and Caputo J G 2018 *Applied Numerical Mathematics* vol 131 (Elsevier) 54–71
[11] Akramov M, Khashimova F and Matrasulov D 2023 *Phys. Lett. A* **457** 128555
[12] Sobirov Z, Matrasulov D, Sawada S and Nakamura K 2011 *Phys. Rev. E* **84** 026609
[13] Sabirov K K, Sobirov Z A, Babajanov D and Matrasulov D U 2013 *Phys.Lett. A* **377** 860
[14] Adami R, Cacciapuoti C, Finco D and Noja D 2011 *Rev. Math. Phys* **23** 4
[15] Noja D 2014 *Philos. Trans. R. Soc.* **372** 20130002
[16] Noja D, Pelinovsky D and Shaikhova G 2015 *Nonlinearity* **28** 2343
[17] Adami R, Cacciapuoti C and Noja D 2016 *J. Diff. Eq.* **260** 7397
[18] Caudrelier V 2015 *Comm. Math. Phys.* **338** 893
[19] Sobirov Z, Babajanov D, Matrasulov D, Nakamura K and Uecker H 2016 *EPL* **115** 50002
[20] Adami R, Serra E and Tilli P 2017 *Commun. Math. Phys.* **352** 387
[21] Kairzhan A and Pelinovsky D E 2018 *J. Phys. A: Math.Theor.* **51** 095203
[22] Sabirov K K, Rakhmanov S, Matrasulov D and Susanto H 2018 *Phys.Lett. A* **382** 1092
[23] Babajanov D, Matyoqubov H and Matrasulov D 2018 *J. Chem. Phys.* **149** 164908
[24] Matrasulov D, Sabirov K, Babajanov D and Susanto H 2020 *EPL* **130** 67002
[25] Yusupov J.R., Matyokubov KhSh, Ehrhardt M and Matrasulov D U 2023 *Phys Lett A* **479** 128928
[26] Matrasulov J. and Sabirov K. 2022 Fokker–Planck equation on metric graphs *Physica A: Statistical Mechanics and its Applications* **608** 128279128279
[27] Giuliano D and Sodano P 2009 *EPL* **88** 17012
[28] Giuliano D and Sodano P 2009 *Nucl. Phys. B* **811** 395
[29] Giuliano D and Sodano P 2010 *Nucl. Phys. B* **837** 153
[30] Giuliano D and Sodano P 2013 *EPL* **103** 57006

[31] Ovchinnikov Yu N and Kresin V Z 2013 *Phys. Rev. B* **88** 214504
[32] Xie F, Ji M and Zhao H 2007 *Chaos Solitons Fractals* **33** 1791–5
[33] Wood D M and Stroud D 1982 *Phys. Rev. B* **25** 1600
[34] Peterson R L and Ekin J W 1988 *Phys. Rev. B* **37** 9848
[35] Fishman R S 1989 *Phys. Rev. B* **39** 7228
[36] Deppe J and Feldman J L 1994 *Phys. Rev. B* **50** 6479
[37] Fazio R and van der Zant H 2001 *Phys. Rep.* **355** 235–334
[38] Abreu L M, Malbouisson A P C and Roditi I 2004 *Physica A* **331** 99–108
[39] Beloborodov I S, Lopatin A V and Vinokur V M 2007 *Revi. Mod. Phys.* **79** 469
[40] Gantmakher V F and Dolgopolov V T 2010 *Phys. Usp.* **53** 1–49
[41] Deutscher G 2021 *J. of Superconduct. and Nov. Magnetism* **34** 1699–703
[42] Li Jun and Chen Yong 2021 *Commun. Theor. Phys.* **73** 015001
[43] Miao Zhengwu and Chen Yong arXiv:2305.07479