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Discrete nonlocal nonlinear Schrödinger equation on graphs: Dynamics of PT-symmetric solitons in discrete networks

M. Akramov [a,*], F. Khashimova [b], D. Matrasulov [c]

[a] National University of Uzbekistan, 4 University Str., 100174 Tashkent, Uzbekistan
[b] Navoi State Mining Institute, 27 Janubiy Str., 210100 Navoi, Uzbekistan
[c] Turin Polytechnic University in Tashkent, 17 Niyazov Str., 100095 Tashkent, Uzbekistan

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ABSTRACT

We consider discrete nonlocal nonlinear Schrödinger equation proposed earlier by Ablowitz and Musslimani, on a branched 1D lattice, presented in terms of the graphs with discrete bonds. The soliton solutions are derived for some simplest (star and tree) graphs. Numerical solutions are obtained for the cases, when the problem does not approve the analytical one. Integrability of the problem is shown by proving existence of infinite number of conservation laws.

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1. Introduction

PT-symmetric nonlocal nonlinear Schrödinger (NNLS) equation was introduced first by Ablowitz and Musslimani in [1]. Remarkable property of NNLS equation is that there are infinitely many conservative quantities and explicit soliton solutions which can be obtained using the inverse scattering approach. The Refs. [1–15] consider integrability, various soliton solutions with their properties and some other features of the NNLS equation over the past few years. Later, in [2], discrete version of NNLS equation was introduced. In particular, it was shown, in analogy with the Ablowitz-Ladik equation, that it also integrable and approved soliton solutions. Discrete nonlinear Schrödinger equations describe wave/soliton propagation in optical waveguide arrays [18], Toda lattices [19], Hirota's nonlinear network [20], etc. Other models for experimental realization of discrete nonlocal nonlinear Schrödinger (DNNLS) equation deal with ferromagnetic materials. Recently the gauge equivalence of DNNLS equation to the unconventional system of coupled Landau-Lifshitz equations has been shown [21]. In particular, a mathematical model, described by coupled Landau-Lifshitz-Hilbert equations, where an additional dissipative Hilbert term is used for implementation of the gain-loss principle is presented. Such model makes the possibility to realize PT-symmetric system described in terms of DNNLS equation in ferromagnetic structures. In [22] the gauge equivalence of DNNLS equation to Heisenberg ferromagnetic equation was shown. The model presented in [22] can be considered as an additional confirmation of the possibility for realization of PT-symmetric systems described by DNNLS equation in ferromagnetic materials. Various versions of discrete NNLS equation and other discrete nonlocal evolution equations have been considered in the Refs. [23–28].
In this paper we extend the study of the Refs. [2,12] to the case of branched 1D domains, modeling these latter in terms of metric graphs. The metric graph can be determined as the set of bonds which are connected to each other according to a some rule. In particular, we address the problem of PT-symmetric discrete nonlocal nonlinear Schrödinger equation on metric graphs with the focus on the exact (soliton) solutions and soliton dynamics. Motivation for the study of soliton dynamics in networks and branched structures comes from the fact that the wave dynamics in these latter is richer and more tunable. One can, e.g. achieve needed wave transport regime by tuning the network architecture or reflection transmission and the nodes (vertices). Another important issue related to modeling of soliton dynamics

* Corresponding author.
E-mail address: mashrabresearcher@gmail.com (M. Akramov).

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in networks is caused by the fact that in practical applications (e.g., in optics and optoelectronics) for signal transfer one often uses optical fiber networks. Therefore studying the soliton propagation in optical networks is of importance from the viewpoint of controlling the signal transfer, reducing the signal losses and optoelectronic device optimization. We note that earlier the soliton dynamics in networks described in terms of nonlinear Schrödinger [29–37], sine-Gordon [38], Bogoliubov de Gennes [39] and nonlinear Dirac [40] equations has attracted much attention. Different aspects on nonlinear evolution equations on networks are considered in the Refs. [41–52]. The model we propose describes soliton propagation in discrete optical waveguide arrays, Toda lattices and Hirota's nonlinear network with self-induced PT-symmetric nonlinearity. The paper is structured as follows. The next section, briefly recalls the DNNLS equation on a line following the Ref. [2]. Section 3 presents formulation and solution of the problem, including analytical solutions, and section 4 presents numerical solutions. In section 5, the integrability of the problem for a discrete star branched network is given. In section 6 we extend the study for tree-like networks. Some concluding remarks are presented in Section 7.

2. Discrete nonlocal nonlinear Schrödinger equation on a line

Discrete version of the Ablowitz-Musslimani nonlocal nonlinear Schrödinger equation can be written as [2]

$$i\frac{dQ_n}{dt} = Q_{n+1} - 2Q_n + Q_{n-1} + Q_n Q^*_{-n}(Q_{n+1} + Q_{n-1}), \tag{1}$$

where n being integer in $(-\infty, +\infty)$. This equation can be considered as PT-symmetric, nonlocal analog of the classical discrete nonlinear Schrödinger equation [53,54], i.e., new and simple reduction of the well-known Ablowitz-Ladik scattering problem. Integrability and soliton solutions of (1) was studied in [2]. In particular, single soliton solution of Eq. (1), obeying the asymptotic condition as $\lim_{n\to\pm\infty} Q_n \to 0$, can be obtained using the inverse scattering method and is given by

$$Q_n(t) = -\frac{(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)e^{i\bar{\varphi}_1}e^{-2i\bar{\omega}_1 t}\bar{z}_1^{2n}}{1 + e^{i(\varphi_1+\bar{\varphi}_1)}e^{2i(\omega_1-\bar{\omega}_1)t}z_1^{-2n}\bar{z}_1^{2n}}, \tag{2}$$

where, $z_1 > 0$, $0 < \bar{z}_1 < 1$, $\omega_1 = (z_1 - z_1^{-1})/2$, $\bar{\omega}_1 = (\bar{z}_1 - \bar{z}_1^{-1})/2$ and φ_1, $\bar{\varphi}_1$ are arbitrary constants. The solution given by Eq. (2) presents a breathing soliton [2]. Another one soliton solution was obtained in the Ref. [16] and given as

$$Q_n(t) = \frac{\varepsilon e^{\kappa n + \omega t}}{1 + \varepsilon^2\theta e^{(\kappa-\kappa^*)n + (\omega+\omega^*)t}}, \tag{3}$$

where $\omega = -2i(\cosh\kappa - 1)$, $\theta = \frac{1}{4}\sinh^{-2}\frac{\kappa-\kappa^*}{2}$ and ε, κ are constant numbers. Third type of the soliton solution was derived in the Ref. [17] by bilinearization-reduction approach can be written as

$$Q_n(t) = \frac{e^{2k_1} - e^{-2k_1^*}}{e^{k_1+k_1^*}(e^{-2k_1 n + 2i\xi_1 t} - e^{-2k_1^* n + 2i\xi_1^* t})}, \tag{4}$$

where $\xi_1 = (e^{2k_1} - 2 + e^{-2k_1})/2$ and k_1 is a constant number.

Existence of infinitely many conservative quantities was also shown in [2]. The first few of them can be written as

$$C_1 = -\sum_{n=-\infty}^{+\infty} Q_n Q^*_{1-n}, \tag{5}$$

$$C_2 = -\sum_{n=-\infty}^{+\infty}\left[Q_n Q^*_{2-n} + \frac{1}{2}(Q_n Q^*_{1-n})^2\right], \tag{6}$$

$$C_3 = \prod_{n=-\infty}^{+\infty}(1 + Q_n Q^*_{-n}). \tag{7}$$

Eq. (1) approves also a Hamiltonian as conserving quantity and non canonical brackets which are given by

$$H = \sum_{n=-\infty}^{+\infty}\left[Q^*_{-n}(Q_{n+1} + Q_{n-1}) - 2\log(1 + Q_n Q^*_{-n})\right], \tag{8}$$

$$\{Q_m, Q^*_{-n}\} = -i(1 + Q_n Q^*_{-n})\delta_{n,m}, \tag{9}$$

$$\{Q_n, Q_m\} = \{Q^*_{-n}, Q^*_{-m}\} = 0. \tag{10}$$

By using last equations Eq. (1) can be written in the form

$$i\frac{dQ_n}{dt} = \{Q_n, H\}. \tag{11}$$

In the next section, the problem studied by Ablowitz and Musslimani in [2] is extended to the case of branched lattices modeled in terms of discrete metric graphs to obtain soliton solutions of DNNLS equation on such lattices using the solution given by Eq. (2).

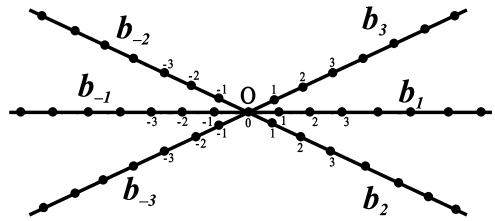

Fig. 1. Star graph with six discrete bonds.

3. PT-symmetric nonlocal solitons in discrete star shaped network

The DNNLS equation on the line considered in the previous section can be extended to the case of branched quasi-one dimensional domains modeled in terms of graphs with discrete bonds. Simplest graph, where such a problem can be considered has a six discrete bonds connected in a single node and called star graph (see, Fig. 1). It represents a branched 1D lattice with six arms. The individual lattice sites in the graph are denoted as $(\pm j, n)$, where $j = 1, 2, 3$ are the bond's number and n corresponds to a lattice site on each bond. Using the metric graph terminology, one can define vertex of the graph, which simply branched site, whose branching point lies at 0 (see, Fig. 1). For the left handed $(-j)$ bonds $n \in b_{-j} = \{0, -1, -2, ...\}$, where $(-j, 0)$ means the branching point. For the right handed (j) bonds $n \in b_j = \{1, 2, 3, ...\}$, where $(j, 1)$ stand for the points nearest to the vertex.

Here we consider the following DNNLS equation on each bond of the star graph presented in Fig. 1 as

$$i\frac{dQ_{\pm j,n}}{dt} = Q_{\pm j,n+1} - 2Q_{\pm j,n} + Q_{\pm j,n-1} + \sqrt{\beta_j\beta_{-j}}Q_{\pm j,n}Q^*_{\mp j,-n}(Q_{\pm j,n+1} + Q_{\pm j,n-1}), \tag{12}$$

where $\beta_{\pm j}$ are the nonlinearity coefficients and $(\pm j, n) \notin \{(-j, 0), (j, 1)\}$. Continuum version of the NNLS equation on such domain was studied in detail recently in [55]. It is important to note that unlike to its local counterpart considered in the Ref. [30], NNLS equation on metric star graph given by Eq. (12), is a system of equations, where coupling occurs between the symmetric components (see, also the Ref. [55]). In addition, due to the nonlocality, it can be written on a graph, who has even number of bonds, which are symmetric with respect to the origin of coordinates. We impose the following asymptotic condition for Eq. (12): $\lim_{n\to-\infty} Q_{-j,n} \to 0$ and $\lim_{n\to+\infty} Q_{j,n} \to 0$. We impose the boundary conditions at the vertex and the sites near the vertex. In case of continuum NNLS equation such boundary conditions can be derived from the energy and the norm conservation laws [55]. However, for discrete evolution equations on graphs derivation of vertex boundary conditions is complicated. Therefore, following the Ref. [30], we explore first the dynamics of solitons in the vicinity of the vertex, i.e. in the sites $(-j, 1), (j, 0)$. Using the same approach as that for Ablowitz-Ladik equation on metric graphs, considered in [30], one can show that at the virtual sites, $(-j, 1), (j, 0)$, Eq. (12) can be obtained from the following equation of motion

$$i\frac{dQ_{\pm j,n}}{dt} = \{Q_{\pm j,n}, H\}, \tag{13}$$

at $(\pm j, n) \notin \{(-j, 0), (j, 1)\}$. Here the non canonical Poisson brackets are determined as [30]

$$\begin{aligned}
&\{Q_{\pm j,m}, Q_{\mp j',-n}\} = -i(1 + \sqrt{\beta_j\beta_{-j}}Q_{\pm j,n}Q^*_{\mp j,-n})\delta_{jj'}\delta_{nm},\\
&\{Q_{\pm j,n}, Q_{\pm j',m}\} = \{Q^*_{\pm j,-n}, Q^*_{\pm j',-m}\} = 0,
\end{aligned} \tag{14}$$

and H is the Hamiltonian for DNNLS equation given by

$$\begin{aligned}
H = \sum_{j=1}^{3}\Bigg[&\sum_{n=0}^{-\infty}\left(Q^*_{j,-n}(Q_{-j,n+1} + Q_{-j,n-1}) - \frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1 + \sqrt{\beta_j\beta_{-j}}Q_{-j,n}Q^*_{j,-n})\right)\\
&+\sum_{n=1}^{+\infty}\left(Q^*_{-j,-n}(Q_{j,n+1} + Q_{j,n-1}) - \frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1 + \sqrt{\beta_j\beta_{-j}}Q_{j,n}Q^*_{-j,-n})\right)\Bigg],
\end{aligned} \tag{15}$$

where at the sites, $(-j, 1), (j, 0)$, following the Ref. [30], we assumed the following relations:

$$\begin{aligned}
&Q_{-j,1} = \alpha_1^{(-j)}Q_{1,1} + \alpha_2^{(-j)}Q_{2,1} + \alpha_3^{(-j)}Q_{3,1},\\
&Q_{j,0} = \alpha_{-1}^{(j)}Q_{-1,0} + \alpha_{-2}^{(j)}Q_{-2,0} + \alpha_{-3}^{(j)}Q_{-3,0}
\end{aligned} \tag{16}$$

with appropriate coefficients, $\alpha_{\pm j}^{(\pm j)}$. Eq. (16) presents the boundary conditions at the "vertex", which are the discrete analogs of the vertex boundary conditions to be imposed in the case of continuum counterpart of our problem considered in [55]. Unlike the continuum case, for discrete graph one needs to impose "vertex" boundary conditions at the "vertex" site.

Explicitly, Eq. (13) at the sites $(-j, 0), (j, 1)$ can be written, respectively as

$$\begin{aligned}
&i\frac{dQ_{-j,0}}{dt} = \alpha_1^{(-j)}Q_{1,1} + \alpha_2^{(-j)}Q_{2,1} + \alpha_3^{(-j)}Q_{3,1} - 2Q_{-j,0} + Q_{-j,-1}\\
&+\sqrt{\beta_j\beta_{-j}}Q_{-j,0}\left(\alpha_{-1}^{(j)}Q^*_{-1,0} + \alpha_{-2}^{(j)}Q^*_{-2,0} + \alpha_{-3}^{(j)}Q^*_{-3,0}\right)\left(\alpha_1^{(-j)}Q_{1,1} + \alpha_2^{(-j)}Q_{2,1} + \alpha_3^{(-j)}Q_{3,1} + Q_{-j,-1}\right),
\end{aligned} \tag{17}$$

$$i\frac{dQ_{j,1}}{dt} = Q_{j,2} - 2Q_{j,1} + \alpha_{-1}^{(j)}Q_{-1,0} + \alpha_{-2}^{(j)}Q_{-2,0} + \alpha_{-3}^{(j)}Q_{-3,0} + \sqrt{\beta_j\beta_{-j}}Q_{j,1}Q_{-j,-1}^{*}\left(Q_{j,2} + \alpha_{-1}^{(j)}Q_{-1,0} + \alpha_{-2}^{(j)}Q_{-2,0} + \alpha_{-3}^{(j)}Q_{-3,0}\right). \tag{18}$$

Furthermore, we suppose that the soliton solution of DNNLS equation (12) on a star graph presented in Fig. 1 is given by

$$Q_{\pm j,n}(t) = \frac{1}{\sqrt{\beta_{\pm j}}}Q_n(t), \tag{19a}$$

where $Q_n(t)$ is the soliton solution of the DNNLS equation on a discrete line, (i.e. on an infinite chain) given by Eq. (1). Substituting the Eq. (2) into Eq. (19a), one can show that the breathing soliton solution of DNNLS equation on the star graph in the following form:

$$Q_{\pm j,n}(t) = -\frac{1}{\sqrt{\beta_{\pm j}}}\frac{(z_1\bar{z}_1)^{-1}(z_1^2 - \bar{z}_1^2)e^{i\bar{\varphi}_1}e^{-2i\tilde{\omega}_1 t}\bar{z}_1^{2n}}{1 + e^{i(\varphi_1+\bar{\varphi}_1)}e^{2i(\omega_1-\tilde{\omega}_1)t}z_1^{-2n}\bar{z}_1^{2n}}. \tag{19b}$$

Comparing Eqs. (1), (17), (18) and using Eq. (19a), one can find two relations between the coefficients $\alpha_{\mp k}^{(\pm j)}$ and $\beta_{\pm j}$:
The first choice is given by

$$\alpha_{\mp k}^{(\pm j)} = \frac{1}{3}\sqrt{\frac{\beta_{\mp k}}{\beta_{\pm j}}}, \tag{20}$$

and the second choice

$$\alpha_{\mp k}^{(\pm j)} = \frac{1}{\sqrt{\beta_{\pm j}\beta_{\mp k}}\left(\frac{1}{\beta_{\pm 1}} + \frac{1}{\beta_{\pm 2}} + \frac{1}{\beta_{\pm 3}}\right)}, \tag{21}$$

where $k = 1, 2, 3$.

Substituting Eq. (21) into Eq. (16), one can obtain the following "quasi-boundary" conditions at the vertex:

$$\sqrt{\beta_{-1}}Q_{-1,1} = \sqrt{\beta_{-2}}Q_{-2,1} = \sqrt{\beta_{-3}}Q_{-3,1},$$
$$\frac{Q_{-1,1}}{\sqrt{\beta_{-1}}} + \frac{Q_{-2,1}}{\sqrt{\beta_{-2}}} + \frac{Q_{-3,1}}{\sqrt{\beta_{-3}}} = \frac{Q_{1,1}}{\sqrt{\beta_1}} + \frac{Q_{2,1}}{\sqrt{\beta_2}} + \frac{Q_{3,1}}{\sqrt{\beta_3}}, \tag{22}$$

and

$$\sqrt{\beta_1}Q_{1,0} = \sqrt{\beta_2}Q_{2,0} = \sqrt{\beta_3}Q_{3,0},$$
$$\frac{Q_{1,0}}{\sqrt{\beta_1}} + \frac{Q_{2,0}}{\sqrt{\beta_2}} + \frac{Q_{3,0}}{\sqrt{\beta_3}} = \frac{Q_{-1,0}}{\sqrt{\beta_{-1}}} + \frac{Q_{-2,0}}{\sqrt{\beta_{-2}}} + \frac{Q_{-3,0}}{\sqrt{\beta_{-3}}}. \tag{23}$$

These quasi-boundary conditions are fulfilled by the solutions (19a), provided the nonlinearity coefficients, $\beta_{\pm j}$ fulfill the constraints given by

$$\frac{1}{\beta_{-1}} + \frac{1}{\beta_{-2}} + \frac{1}{\beta_{-3}} = \frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}. \tag{24}$$

We note that the sum rule in Eq. (24) ensures that function given by Eq. (19a) is the solution of the DNNLS equation on graph, given by Eqs. (12), (22) and (23). In other words, in case, when the nonlinearity coefficients, $\beta_{\pm j}$ fulfill the sum rule in Eq. (24), solution of Eq. (1), presented in the form given by Eq. (19a), will be the solution of the DNNLS equation on graph, given by Eqs. (12), (22) and (23). Also, it is important to mention that the sum rule in Eq. (24) on the basis of the first choice, i.e. Eq. (21), although both choices ensure integrability of the problem. However, unlike the first choice, the second choice allows to obtain constraints for integrability in terms of nonlinearity coefficients, $\beta_{\pm j}$.

4. Numerical results

The solution given by Eq. (19a) presents a soliton solution of Eq. (12), that fulfills the vertex boundary conditions (22) and (23), provided constraints given by Eq. (24) are valid. In other words, explicit soliton solution of the problem given by Eqs. (12), (22) and (23) can be written only for those values of $\beta_{\pm j}$, which fulfill the sum rule in Eq. (24). In general case, when the sum rule is not fulfilled, one needs to solve Eq. (12) numerically. In Fig. 2 contour plot of the (breathing) soliton solution of Eq. (12) for the quasi-boundary conditions given by Eqs. (22) and (23) is presented. The plot is obtained using the analytical formula (19a) for the values of parameters $\beta_{-1} = 1$, $\beta_1 = 1.15$, $\beta_{-2} = 2.19$, $\beta_2 = 1.92$, $\beta_{-3} = 2.42$, $\beta_3 = 2.09$, which fulfill the sum rule (24). Fig. 3 presents numerically obtained traveling wave solution of DNNLS equation on graph, (12) for the case, when the sum rule given by Eq. (24) is fulfilled. One can observe from this plot that transmission of solitons through the vertex occurs without reflection, unlike the Fig. 4, where numerically obtained traveling wave solutions are plotted for the values of $\beta_{\pm j}$, which do not fulfill constraint (24). Reflection of soliton at the branching area can be clearly seen from these plots. In numerical solution of Eq. (12) the initial conditions are chosen on the b_{-1} and b_1 bonds, in the form of Gaussian wave packet given by

$$Q_{\pm 1,n} = \frac{A}{\sqrt{\beta_{\pm 1}\sigma\sqrt{2\pi}}}\exp\left[\pm ik_0 n - \left(\frac{n - N_{\pm}}{2\sigma^2}\right)^2\right], \tag{25}$$

where $A = 0.1$, $k_0 = 1$, $\sigma = 8$ and $N_{\pm} = \pm 50$ are constants.

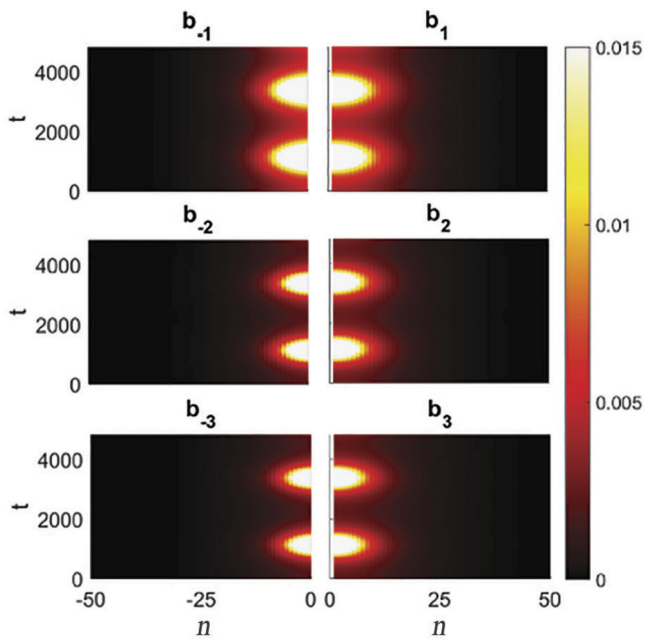

Fig. 2. Contour plot of the breathing soliton solution of DNNLS equation (12) on metric star graph given by Eq. (19a). The nonlinearity coefficients are chosen as $\beta_{-1} = 1$, $\beta_1 = 1.15$, $\beta_{-2} = 2.19$, $\beta_2 = 1.92$, $\beta_{-3} = 2.42$, $\beta_3 = 2.09$, i.e. the sum rule in Eq. (24) is fulfilled. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

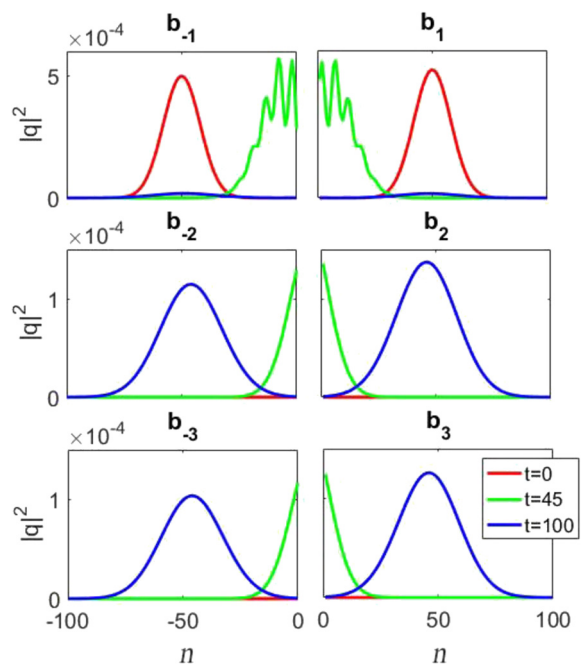

Fig. 3. Numerically obtained traveling wave solution of DNNLS equation on the star graph when the sum rule, (24) is fulfilled. The nonlinearity coefficients are chosen as $\beta_{-1} = 1$, $\beta_1 = 1.15$, $\beta_{-2} = 2.19$, $\beta_2 = 1.91$, $\beta_{-3} = 2.42$, $\beta_3 = 2.09$. The initial conditions are given on the two symmetric bonds b_{-1} and b_1.

5. Conservation laws and integrability

Integrability, i.e., the existence of infinitely many conservative quantities of DNNLS equation on the star graph can be proved similarly to that in [2]. Similar thing is possible for DNNLS equation on graph, too. For Eq. (12), fundamental conservative quantities, and the energy, (for star graph, in Fig. 1) can be written as, respectively

$$C_1 = -\sum_{j=1}^{3}\left[\sum_{n=0}^{-\infty} Q_{-j,n} Q^*_{j,1-n} + \sum_{n=1}^{+\infty} Q_{j,n} Q^*_{-j,1-n}\right], \tag{26}$$

$$C_2 = -\sum_{j=1}^{3}\left[\sum_{n=0}^{-\infty}\left(Q_{-j,n} Q^*_{j,2-n} + \frac{\sqrt{\beta_{-j}\beta_j}}{2}(Q_{-j,n} Q^*_{j,1-n})^2\right) + \sum_{n=1}^{+\infty}\left(Q_{j,n} Q^*_{-j,2-n} + \frac{\sqrt{\beta_{-j}\beta_j}}{2}(Q_{j,n} Q^*_{-j,1-n})^2\right)\right], \tag{27}$$

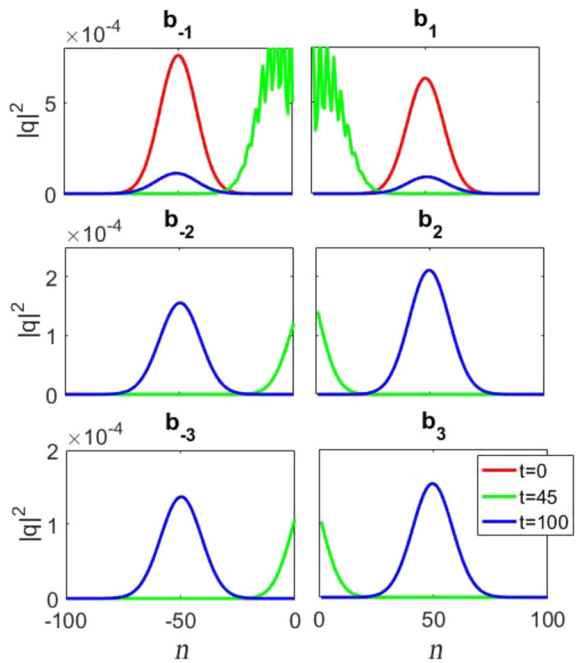

Fig. 4. Numerically obtained traveling wave solution of DNNLS equation on the star graph when the sum rule, (24) is broken. The nonlinearity coefficients are chosen as $\beta_{-1} = 0.65$, $\beta_1 = 0.79$, $\beta_{-2} = 2.7$, $\beta_2 = 2.09$, $\beta_{-3} = 3.06$, $\beta_3 = 2.87$. The initial conditions are given on the two symmetric bonds b_{-1} and b_1.

$$C_3 = \prod_{j=1}^{3}\left[\prod_{n=0}^{-\infty}(1+\sqrt{\beta_{-j}\beta_j}Q_{-j,n}Q^*_{j,-n})\cdot\prod_{n=1}^{+\infty}(1+\sqrt{\beta_{-j}\beta_j}Q_{j,n}Q^*_{-j,-n})\right]. \tag{28}$$

$$\begin{aligned}
E = &\sum_{j=1}^{3}\Bigg[\sum_{n=-1}^{-\infty}\left(Q^*_{j,-n}(Q_{-j,n+1}+Q_{-j,n-1}) - \frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1+\sqrt{\beta_j\beta_{-j}}Q_{-j,n}Q^*_{j,-n})\right)+\\
&\sum_{n=2}^{+\infty}\left(Q^*_{-j,-n}(Q_{j,n+1}+Q_{j,n-1}) - \frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1+\sqrt{\beta_j\beta_{-j}}Q_{j,n}Q^*_{-j,-n})\right)+\\
&(\alpha^{(j)}_{-1}Q^*_{-1,0}+\alpha^{(j)}_{-2}Q^*_{-2,0}+\alpha^{(j)}_{-3}Q^*_{-3,0})(\alpha^{(-j)}_{1}Q_{1,1}+\alpha^{(-j)}_{2}Q_{2,1}+\alpha^{(-j)}_{3}Q_{3,1}+Q_{-j,-1})-\\
&\frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1+\sqrt{\beta_j\beta_{-j}}Q_{-j,0}(\alpha^{(j)}_{-1}Q^*_{-1,0}+\alpha^{(j)}_{-2}Q^*_{-2,0}+\alpha^{(j)}_{-3}Q^*_{-3,0}))+\\
&Q^*_{-j,-1}(Q_{j,2}+\alpha^{(j)}_{-1}Q_{-1,0}+\alpha^{(j)}_{-2}Q_{-2,0}+\alpha^{(j)}_{-3}Q_{-3,0}) - \frac{2}{\sqrt{\beta_j\beta_{-j}}}\log(1+\sqrt{\beta_j\beta_{-j}}Q_{j,1}Q^*_{-j,-1})\Bigg].
\end{aligned} \tag{29}$$

Conservation laws for these quantities are written in terms of their time derivatives as $\frac{d}{dt}C_1 = 0$, $\frac{d}{dt}C_2 = 0$, $\frac{d}{dt}C_3 = 0$ and $\frac{d}{dt}E = 0$. Existence of infinitely many conserving quantities can be shown similarly to that for DNNLS equation on the line studied in detail in [2].

6. Extension to a tree graph

DNNLS equation on the metric star graph can be extended to other more complicated branched topologies. In this section, we will demonstrate this for the tree graph (see, Fig. 5). The graph consists of 10 semi-infinite and 4 finite bonds. These latter are determined as follows: $b_{-1}, b_{-1mn} = \{0, -1, -2, -3, ...\}$, $b_{-1m} = \{0, -1, -2, ..., -N_m\}$, $b_1, b_{1mn} = \{1, 2, 3, ...\}$, $b_{1m} = \{1, 2, 3, ..., N_m\}$, where $m, n = 1, 2$.

DNNLS equation on each bond of the tree graph can be written as

$$i\frac{dQ_{\pm e,n}}{dt} = Q_{\pm e,n+1} - 2Q_{\pm e,n} + Q_{\pm e,n-1} + \sqrt{\beta_e\beta_{-e}}Q_{\pm e,n}Q^*_{\mp e,-n}(Q_{\pm e,n+1}+Q_{\pm e,n-1}), \tag{30}$$

where $e = 1, 1m, 1mn$. We note that Eq. (30) is written at the sites, which do not touch the vertices (branching points), i.e., the sites in nearest vicinity of vertices. Similarly to that for the star graph, we impose the following conditions on four symmetric branching points:

$$\begin{aligned}
&Q_{-1mn,1} = \alpha^{(-1mn)}_{-1m}Q_{-1m,-N_m},\\
&Q_{-1m,-N_m+1} = \alpha^{(-1m)}_{-1m1}Q_{-1m1,0} + \alpha^{(-1m)}_{-1m2}Q_{-1m2,0},\\
&Q_{1mn,0} = \alpha^{(1mn)}_{1m}Q_{1m,N_m},\\
&Q_{1m,N_m+1} = \alpha^{(1m)}_{1m1}Q_{1m1,1} + \alpha^{(1m)}_{1m2}Q_{1m2,1}.
\end{aligned} \tag{31}$$

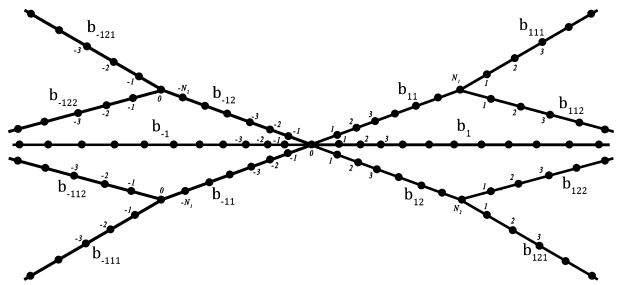

Fig. 5. Sketch of a tree graph.

Soliton solutions of DNNLS equation on each branch of the tree graph can be expressed in terms of the solution of DNNLS equation on the line as

$$Q_{\pm e,n}(t) = \frac{1}{\sqrt{\beta_{\pm e}}} Q_{n+s_{\pm e}}(t), \tag{32}$$

where $s_{\pm 1} = s_{\pm 1m} = n_0$, $s_{\pm 1mn} = n_0 \pm N_m$ and n_0 is the center of soliton. Requiring fulfilling the conditions (31), by the solution (32), one obtains the following relations between coefficients, $\alpha_{\pm e}^{(\mp e)}$ and nonlinearity parameters, $\beta_{\pm e}$:

$$\begin{aligned} \alpha_{\pm 1mn}^{(\pm 1m)} &= \sqrt{\frac{\beta_{\pm 1m}}{\beta_{\pm 1mn}}}, \\ \alpha_{\pm 1m}^{(\pm 1mn)} &= \frac{1}{\sqrt{\beta_{\pm 1m}\beta_{\pm 1mn}}\left(\frac{1}{\beta_{\pm 1m1}} + \frac{1}{\beta_{\pm 1m2}}\right)}. \end{aligned} \tag{33}$$

Substituting of Eq. (33) into Eq. (31) lead to the following "quasi-boundary" conditions at the vertices:

$$\begin{aligned} &\sqrt{\beta_{-1m1}}\, Q_{-1m1,1} = \sqrt{\beta_{-1m2}}\, Q_{-1m2,1}, \\ &\frac{Q_{-1m1,1}}{\sqrt{\beta_{-1m1}}} + \frac{Q_{-1m2,1}}{\sqrt{\beta_{-1m2}}} = \frac{Q_{-1m,-N_m}}{\sqrt{\beta_{-1m}}}, \\ &\frac{Q_{-1m,-N_m+1}}{\sqrt{\beta_{-1m}}} = \frac{Q_{-1m1,0}}{\sqrt{\beta_{-1m1}}} + \frac{Q_{-1m2,0}}{\sqrt{\beta_{-1m2}}}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\sqrt{\beta_{1m1}}\, Q_{1m1,0} = \sqrt{\beta_{1m2}}\, Q_{1m2,0}, \\ &\frac{Q_{1m1,0}}{\sqrt{\beta_{1m1}}} + \frac{Q_{1m2,0}}{\sqrt{\beta_{1m2}}} = \frac{Q_{1m,-N_m}}{\sqrt{\beta_{1m}}}, \\ &\frac{Q_{1m,-N_m+1}}{\sqrt{\beta_{1m}}} = \frac{Q_{1m1,1}}{\sqrt{\beta_{1m1}}} + \frac{Q_{1m2,1}}{\sqrt{\beta_{1m2}}}. \end{aligned} \tag{35}$$

The above quasi-boundary conditions fulfill by the solution Eq. (32), if the following constraint in terms of $\beta_{\pm e}$ to be hold

$$\frac{1}{\beta_{\pm 1m}} = \frac{1}{\beta_{\pm 1m1}} + \frac{1}{\beta_{\pm 1m2}}. \tag{36}$$

The sum rule given by Eq. (36) ensures integrability of DNNLS equation (30) on tree graph domain presented in Fig. 5 and approving soliton solutions given by Eq. (32). We note that the above utilized approach can be applied for solving DNNLS equation on graphs of arbitrary topology, provided graph has two for outgoing semi-infinite bonds.

7. Conclusions

In this paper, we extended the problem of nonlocal soliton dynamics which can be determined in terms of Ablowitz-Musslimani PT-symmetric discrete nonlocal nonlinear Schrödinger equation on the metric graphs with discrete bonds. Integrability of the problem on the star shaped graph is shown. Analytic expressions for soliton solutions of DNNLS equation on branched discrete lattices are obtained for the case given in terms of constraints. These latter are presented in terms of simple sum rules (24). Numerical solutions that when the constraint (24) was fulfilled, the reflectionless transmission of solitons through the vertex (see, Fig. 3) occurs, while for the case, when constraint is broken, reflection of the soliton at the vertex can be observed (see, Fig. 4). The model proposed in the paper can be used for describing soliton dynamics in discrete waveguide networks such as, e.g., branched Hirota lattices, discrete optical fiber arrays networks, where each branch has self-induced gain-loss. Finally, we would like to note the difference of the above model from that described in terms of local NLS equation considered earlier in [30]. It comes from the fact that unlike to its local counterpart, DNNLS equation on graphs cannot be formulated on primary (basic) star graph, i.e. on simple Y-junction. One needs to have (in case of a star graph) even number of bonds, symmetrically located with respect to the vertex, or (in case of arbitrary graph) the number of incoming semi-infinite bonds should be equal to that of outgoing bonds. This is a consequence of the nonlocality of the considered NLS equation. Another consequence is the fact that unlike local NLSE on graphs, the nonlocal counterpart has coupled components, i.e. it is a system of NLS equations where three components are coupled to each other (j-component is coupled to $-j$ component).

CRediT authorship contribution statement

M. Akramov: Visualization, Formal analysis. **F. Khashimova:** Software. **D. Matrasulov:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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