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Letter

Second-harmonic generation in branched optical waveguides: Metric graphs based approach

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ABSTRACT

We consider the second-harmonic generation in branched optical waveguides, by modeling the process in terms of the three-wave interaction system on metric graphs. The vertex boundary conditions are derived from the conservation laws. Soliton solutions of the three-wave interaction system are obtained for the case of star- and tree-branched networks.

1. Introduction

Nonlinear optical phenomena occurring in low-dimensional optical materials have big potential for practical application in modern optics and optoelectronics. Controlling of these processes and tuning functional properties of such materials is of importance for device optimization and performance improving. An important nonlinear optical effect having application in laser generation and signal transfer in modern technology is the optical second harmonic generation (SHG). The effect implies conversion of the frequency or amplitude of the incoming (fundamental) electromagnetic wave in its passing through (e.g., Kerr) media due to the nonlinearity of the interaction. In particular, the effect can cause doubling of the fundamental wave's frequency. During recent few decades optical SHG attracted much attention in different contexts (see e.g., Refs. [1–22] for review of the progress in the topic). The SHG in the form of solitons was studied in [2–11]. Second harmonic generation caused by three (optical) nonlinear waves was considered in [13–17]. Different versions for experimental realizations of SHG was studied in [12].

In this paper, we consider SHG in soliton regime in branched optical wave-guides, e.g. 1D optical fibers connected to each other at a single junction. In particular, we consider the interaction of three waves. For modeling SHG in such a structure we use a metric graph based approach, i.e. we solve the system of nonlinear evolution equations describing optical SHG process, in a domain called metric graph. The latter represents system of 1D wires, connected to each other at the nodes (vertices) according to a rule, called topology of a graph. We obtain exact solution of the problem for a special case, given by in terms of certain constraints in the form of a simple sum rule.

We note that the soliton dynamics described in terms of the evolution equations on metric graphs has been the subject for extensive research during the past two decades [23–28].

The motivation for the study of the SHG in branched optical waveguides comes from the fact that in most of the practical applications in optoelectronics signal propagation occurs in branched optical structures by the solitons as signal carriers. This makes important the problem of soliton propagation in low-dimensional branched structures.

This paper is organized as follows. In the next section, we briefly recall the SHG problem in the unbranched optical waveguide. Section 3 presents a model for SHG in branched waveguides in the form of solitons in the three-wave interaction. Analysis of the soliton dynamics and second harmonic generation in branched waveguides are presented in section 4. Finally, section 5 presents some concluding results.

2. Second-harmonic generation in unbranched optical waveguide

Here, following the Ref. [13], we briefly recall the second harmonic generation in unbranched waveguide, caused by the interaction of three nonlinear waves. In [13] analytical soliton solutions of the nonlinear three-wave interaction equations are obtained. It was shown that the generated pulses have clean profiles with second harmonics sharper

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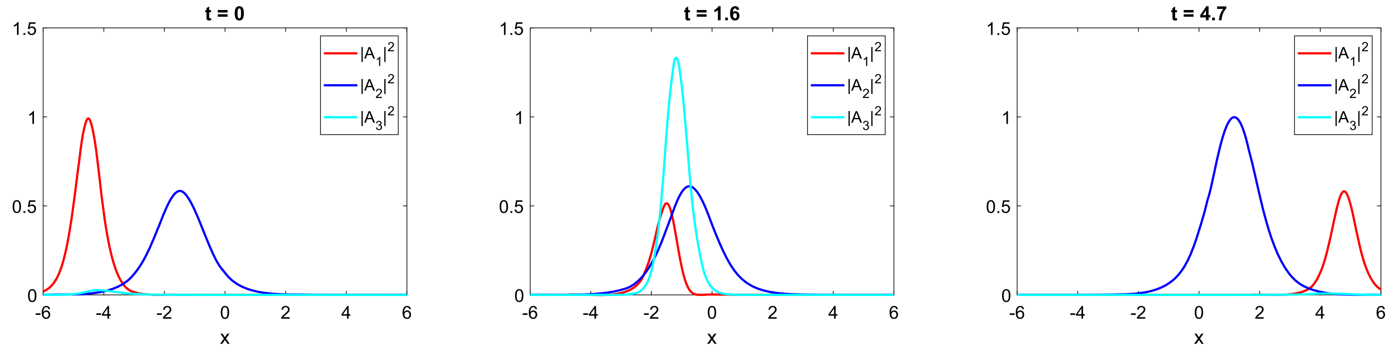

Fig. 1. Profiles of the fundamental waves and the generation of second-harmonic at different time moments in unbranched optical waveguide (plotted using solutions in Eq. (3)). The parameters of the solution are $A_{10} = A_{20} = 1$, $\upsilon_1 = 2$, $\upsilon_2 = 0.5$, $\upsilon_3 = 1.9$, $\sigma = 0.5$.

than the initial fundamental ones. Also, it was found that the fundamental waves exhibit compression in addition to the second harmonics. The system is described in terms of coupled nonlinear evolution equations given by [13]

$$\begin{aligned}
&\frac{1}{\upsilon_1}\partial_t A_1 + \partial_x A_1 = i\sigma_1 A_2^* A_3 \exp(i\Delta k z),\\
&\frac{1}{\upsilon_2}\partial_t A_2 + \partial_x A_2 = i\sigma_2 A_1^* A_3 \exp(i\Delta k z),\\
&\frac{1}{\upsilon_3}\partial_t A_3 + \partial_x A_3 = i\sigma_3 A_1 A_2 \exp(-i\Delta k z),
\end{aligned} \tag{1}$$

where A_n $(n = 1, 2, 3)$ is envelope function of the electric field $E_n(t, z) = A_n(t, z)\exp[i(\omega_n t - k_n z)]$, ω_n is frequency, υ_n is the velocity, k is wave number, σ_n is the nonlinear coupling coefficient, $\Delta k = k_3 - k_1 - k_2$ is phase velocity mismatch. Here the amplitudes A_1, A_2 correspond to the fundamental waves and A_3 to the second harmonic pulse.

It was shown in [4] that the three wave interaction equation is integrable, i.e. has an infinite number of conservation laws. Some important conserving quantities can be written as

$$\begin{aligned}
&C_1 = \int_{-\infty}^{+\infty} \left(\frac{|A_1|^2}{\upsilon_1\sigma_1} - \frac{|A_2|^2}{\upsilon_2\sigma_2} \right) dx,\\
&C_2 = \int_{-\infty}^{+\infty} \left(\frac{|A_1|^2}{\upsilon_1\sigma_1} + \frac{|A_3|^2}{\upsilon_3\sigma_3} \right) dx,\\
&C_3 = \int_{-\infty}^{+\infty} \left(\frac{|A_2|^2}{\upsilon_2\sigma_2} + \frac{|A_3|^2}{\upsilon_3\sigma_3} \right) dx.
\end{aligned} \tag{2}$$

In the following, we assume that $\omega_1 = \omega_2 = \omega_3/2 = \omega$, $\sigma_1 = \sigma_2 = \sigma_3/2 = \sigma > 0$, $\Delta k = 0$. Exact solution of three wave interaction equation (1) can be written as Ref. [13]

$$\begin{aligned}
&A_1(x,t) = \frac{2A_{10}}{D(\xi,\eta)}\left(e^{\xi} - \alpha e^{-\xi}\right),\\
&A_2(x,t) = \frac{2A_{20}}{D(\xi,\eta)}\left(e^{\eta} + \alpha e^{-\eta}\right),\\
&A_3(x,t) = \frac{4iA_{10}A_{20}\sqrt{2\nu_{1,2}}}{D(\xi,\eta)\beta},
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&\xi = \sigma A_{20}\left(\frac{2}{\nu_{1,2}\nu_{2,3}}\right)^{1/2}\left(t - x/\upsilon_2 - T\right),\\
&\eta = \sigma A_{10}\left(\frac{2}{\nu_{1,2}\nu_{1,3}}\right)^{1/2}\left(t - x/\upsilon_1 + T\right),
\end{aligned} \tag{4}$$

where the free parameters A_{10} and A_{20}, $2T$ are the time separation between the waves at $x = 0$. The other parameters in Eq. (3) are given by

$$\begin{aligned}
&\nu_{1,2} = \upsilon_2^{-1} - \upsilon_1^{-1},\\
&\nu_{1,3} = \upsilon_3^{-1} - \upsilon_1^{-1},\\
&\nu_{2,3} = \upsilon_2^{-1} - \upsilon_3^{-1},\\
&\alpha = (A_{20}\sqrt{\nu_{2,3}} - A_{10}\sqrt{\nu_{1,3}})/\beta,\\
&\beta = A_{20}\sqrt{\nu_{2,3}} - A_{10}\sqrt{\nu_{1,3}},\\
&\gamma = 4A_{10}A_{20}\sqrt{\nu_{1,3}\nu_{2,3}}/\beta^2,\\
&D(\xi,\eta) = 4\cosh(\xi)\cosh(\eta) + \gamma e^{-\xi-\eta}.
\end{aligned}$$

The initial profiles of the fundamental pulses before interaction can be written as [13]

$$A_n(x,t) = A_{n0}\operatorname{sech}\left[\frac{\sqrt{2}\sigma a_{n0}}{\sqrt{\nu_{1,2}\nu_{n,3}}}\left(t - \frac{x}{\upsilon_n} + (-1)^n T\right)\right], \tag{5}$$

where $n = 1, 2$.

In Fig. 1 profiles of the (square modules) amplitudes of fundamental (A_1 and A_2) and (generated) second harmonic (A_3) at different time moments (for the following choice of the parameters: $\upsilon_1 = 2$, $\upsilon_2 = 0.5$, $\upsilon_3 = 1.9$, $\sigma = 0.5$, $A_{10} = A_{20} = 1$, $T = -1.8$). As it can be seen from the plot, the duration of the second harmonics pulse is much shorter than that of fundamental ones, while its intensity is much higher. As time elapses, the amplitude of the second harmonic decays and becomes zero that leads to the situation, when two (non-interacting) fundamental waves remain. The shape of these two pulses is the same as that before the interaction. In the next section, we extend similar study to the case of a branched optical waveguide, i.e. waveguide networks, by considering star- and tree-branched cases.

3. Three-wave interaction in optical waveguide networks

3.1. Star branched waveguide

Here we will apply the above approach to the study of SHG in more complicated materials, i.e. optical waveguide networks. We assume that the waveguide is thin enough so that one can ignore by transfers motion. Then the wave motion can be considered as one-dimensional and one does not need to derive the three-wave interaction system from the 3D Maxwell equations using factorization of variables. Therefore, the second harmonic generation in such approach can be described in terms of the three-wave interaction system on metric graphs. The latter are the one-dimensional wires connected to each other at the vertices (branching points) according to the rule called topology of a graph. The advantage of modeling of branched structures in terms of metric graphs they greatly facilitate the mathematical problem used for description of

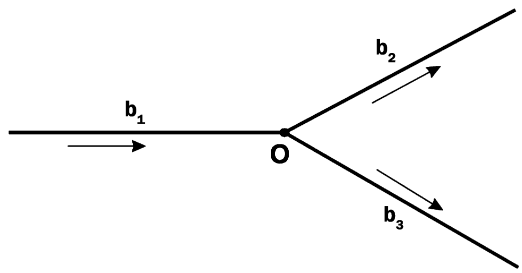

Fig. 2. Basic star graph.

a system by reducing the description into the one-dimensional evolution equations. In such approach, an evolution equation is written on each branch (bond) of a graph and related to each other through the vertex boundary conditions. Unlike their unbranched counterparts, branched nonlinear evolution in such systems strongly depends on the branching topology of a structure that provides more tools for tuning the evolution.

Thus the model we propose consists of a branched structure of optical waveguides interacting with nonlinear optical waves. In such model interaction of the waves occurs in a star-branched metric graph presented in Fig. 2. For each bond of the star graph, we assign a coordinate, for b_1 bond we fix $x_1 \in (-\infty, 0]$ and for $b_{2,3}$ bonds $x_{2,3} \in [0, +\infty)$. The origin of the coordinate is chosen at the branching point, i.e., the vertex.

We introduce the fundamental $A_1^{(j)}(x_j,t)$, $A_2^{(j)}(x_j,t)$ and SH $A_3^{(j)}(x_j,t)$ wavefunctions on each bond of the graph in terms of the wavefunctions $A_n(x,t)$ on a line as (see Ref. [24] for more details)

$$A_n^{(j)}(x_j,t) = \frac{A_n(x,t)}{\beta_j}, \tag{6}$$

where β_j $(j = 1,2,3)$ is the bond dependent parameter.

In the following for shorthand notation we use $A_n^{(j)}(x,t)$ instead of $A_n^{(j)}(x_j,t)$. Then the system of nonlinear evolution equations describing the SHG process in such a structure can be written on each bond of the (star) graph as

$$\begin{aligned}
&\frac{1}{v_1}\partial_t A_1^{(j)} + \partial_x A_1^{(j)} = i\beta_j\sigma_1 A_2^{(j)*}A_3^{(j)},\\
&\frac{1}{v_2}\partial_t A_2^{(j)} + \partial_x A_2^{(j)} = i\beta_j\sigma_2 A_1^{(j)*}A_3^{(j)},\\
&\frac{1}{v_3}\partial_t A_3^{(j)} + \partial_x A_3^{(j)} = i\beta_j\sigma_3 A_1^{(j)}A_2^{(j)}.
\end{aligned} \tag{7}$$

To solve this system of equations one needs to impose the boundary conditions at the branching point. To be physically relevant, such boundary conditions need to be derived from conservation laws. In case of branched waveguide, the counterpart of the conservative quantities in Eqs. (2) can be written as

$$\begin{aligned}
C_1 &= \sum_{j=1}^{3}\int_{b_j}\left(\frac{|A_1^{(j)}|^2}{v_1\sigma_1} - \frac{|A_2^{(j)}|^2}{v_2\sigma_2}\right)dx,\\
C_2 &= \sum_{j=1}^{3}\int_{b_j}\left(\frac{|A_1^{(j)}|^2}{v_1\sigma_1} + \frac{|A_3^{(j)}|^2}{v_3\sigma_3}\right)dx,\\
C_3 &= \sum_{j=1}^{3}\int_{b_j}\left(\frac{|A_2^{(j)}|^2}{v_2\sigma_2} + \frac{|A_3^{(j)}|^2}{v_3\sigma_3}\right)dx.
\end{aligned} \tag{8}$$

Using explicit expressions for the time derivatives of C_n which are given as

$$\frac{dC_1}{dt} = -\sum_{j=1}^{3}\left[\frac{|A_1^{(j)}|^2}{\sigma_1} - \frac{|A_2^{(j)}|^2}{\sigma_2}\right]\Big|_{b_j},$$

$$\frac{dC_2}{dt} = -\sum_{j=1}^{3}\left[\frac{|A_1^{(j)}|^2}{\sigma_1} + \frac{|A_3^{(j)}|^2}{\sigma_3}\right]\Big|_{b_j},$$

$$\frac{dC_3}{dt} = -\sum_{j=1}^{3}\left[\frac{|A_2^{(j)}|^2}{\sigma_2} + \frac{|A_3^{(j)}|^2}{\sigma_3}\right]\Big|_{b_j}.$$

From the conservation laws

$$\frac{dC_n}{dt} = 0, \quad n = 1,2,3,$$

we get

$$|A_n^{(1)}|^2\Big|_{x=0} = |A_n^{(2)}|^2\Big|_{x=0} + |A_n^{(3)}|^2\Big|_{x=0}. \tag{9}$$

One can derive the set of the vertex boundary conditions by linearizing the above equation (9):

$$\alpha_1 A_n^{(1)}\Big|_{x=0} = \alpha_2 A_n^{(2)}\Big|_{x=0} = \alpha_3 A_n^{(3)}\Big|_{x=0}, \tag{10}$$

$$\frac{1}{\alpha_1}\partial_x A_n^{(1)}\Big|_{x=0} = \frac{1}{\alpha_2}\partial_x A_n^{(2)}\Big|_{x=0} + \frac{1}{\alpha_3}\partial_x A_n^{(3)}\Big|_{x=0}, \tag{11}$$

where α_j is the new bond dependent parameter which will be defined later. The solution of Eq. (7) can be written in terms of the solution of three interaction equation in Eq. (3) as in the Eq. (6).

Requiring that $A_n^{(j)}(x,t)$ should fulfill the vertex boundary conditions (9) and (10), we obtain the following constraints by for the parameters, β_j:

$$\frac{\alpha_j}{\alpha_1} = \frac{\beta_j}{\beta_1}, \quad \frac{1}{\beta_1^2} = \frac{1}{\beta_2^2} + \frac{1}{\beta_3^2}. \tag{12}$$

It is important to note that fulfilling the constraints Eq. (12) by choosing the proper values of β_j's allows one to construct soliton solution of the problem given by Eqs. (7) and (10). However, these constraints do not guarantee integrability of the problem, as the perturbations violating the reduction can still lead to non-integrable effects such as, e.g., back-scattering and drift instability (see. e.g., Refs. [29,30] for details).

3.2. Extension to tree branched networks

The above approach for solving three-wave interaction system of equations on the star graph, can be extended to other branching topologies, too. Here we demonstrate this by applying it to tree-branched optical waveguides. Consider tree graph has 4 semi-infinite and 2 finite bonds, presented in Fig. 4. The coordinates of each bond are assigned as follows: $x_1 \in (-\infty, 0]$ for b_1, $x_{1i} \in [0, L_{1i}]$ for b_{1i} and $x_{1ij} \in [0, \infty)$ for b_{1ij} bonds, where $i, j = 1, 2$ and L_{1i} is length of bond b_{1i}. Eqs. (7), describing the second harmonic generation in tree-branched waveguide are written on each of such graph:

$$\begin{aligned}
&\frac{1}{v_1}\partial_t A_1^{(e)} + \partial_x A_1^{(e)} = i\beta_e\sigma_1 A_2^{(e)*}A_3^{(e)},\\
&\frac{1}{v_2}\partial_t A_2^{(e)} + \partial_x A_2^{(e)} = i\beta_e\sigma_2 A_1^{(e)*}A_3^{(e)},\\
&\frac{1}{v_3}\partial_t A_3^{(e)} + \partial_x A_3^{(e)} = i\beta_e\sigma_3 A_1^{(e)}A_2^{(e)},
\end{aligned} \tag{13}$$

where $e = 1, 1i, 1ij$.

Similarly to that star graph, the vertex boundary conditions can be derived from the conservation laws that leads to

$$\begin{aligned}
&\alpha_1 A_n^{(1)}\Big|_{x=0} = \alpha_{11} A_n^{(11)}\Big|_{x=0} = \alpha_{12} A_n^{(12)}\Big|_{x=0},\\
&\frac{1}{\alpha_1}\partial_x A_n^{(1)}\Big|_{x=0} = \frac{1}{\alpha_{11}}\partial_x A_n^{(11)}\Big|_{x=0} + \frac{1}{\alpha_{12}}\partial_x A_n^{(12)}\Big|_{x=0},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\alpha_{1i} A_n^{(1i)}\Big|_{x=L_{1i}} = \alpha_{1ij} A_n^{(1ij)}\Big|_{x=0} = \alpha_{1ij} A_n^{(1ij)}\Big|_{x=0},\\
&\frac{1}{\alpha_{1i}}\partial_x A_n^{(1i)}\Big|_{x=L_{1i}} = \frac{1}{\alpha_{1ij}}\partial_x A_n^{(1ij)}\Big|_{x=0} + \frac{1}{\alpha_{1ij}}\partial_x A_n^{(1ij)}\Big|_{x=0}.
\end{aligned} \tag{15}$$

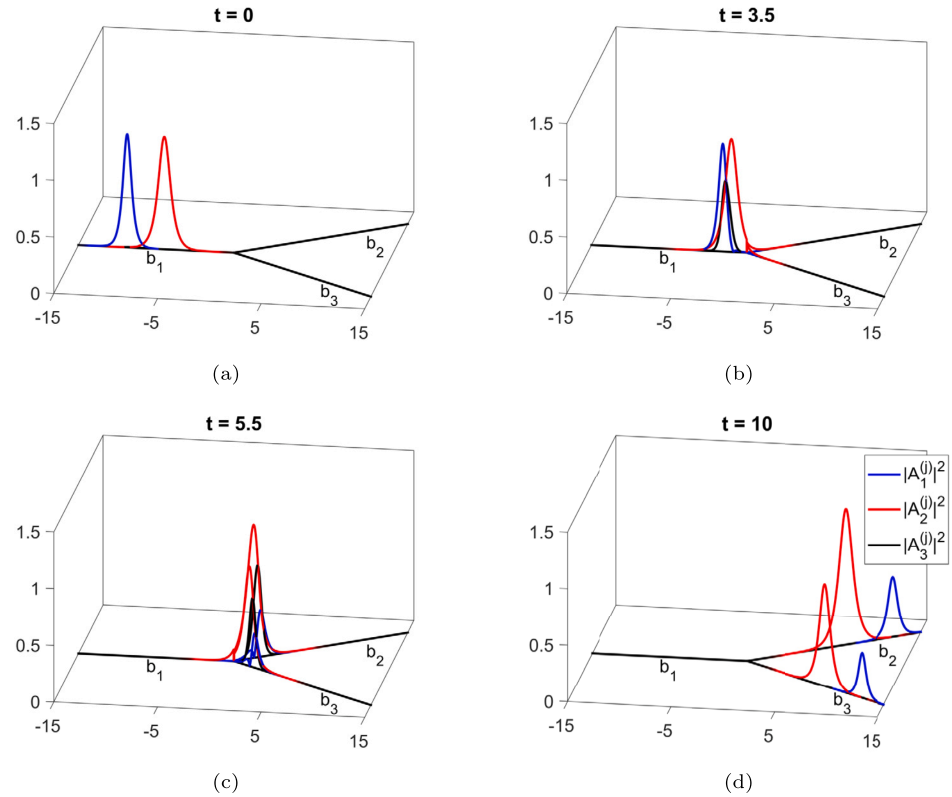

Fig. 3. Profiles the soliton generated as second harmonic in three wave interaction on the star-branched waveguide for the case, when the sum rule in Eq. (12) is fulfilled for the following values of β_j: $\beta_1 = 1$, $\beta_2 = 5/4$, $\beta_3 = 5/3$.

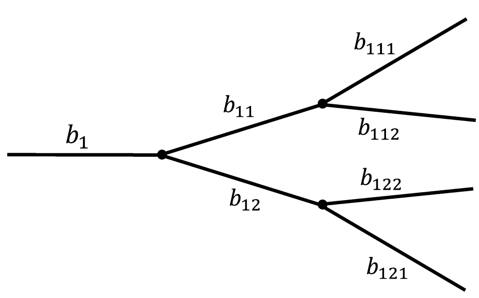

Fig. 4. Tree graph.

Assuming that $\beta_j i$ fulfill the following sum rules

$$\frac{\alpha_{1i}}{\alpha_1} = \frac{\beta_{1i}}{\beta_1}, \quad \frac{1}{\beta_1^2} = \frac{1}{\beta_{11}^2} + \frac{1}{\beta_{12}^2},$$
$$\frac{\alpha_{1ij}}{\alpha_{1i}} = \frac{\beta_{1ij}}{\beta_{1i}}, \quad \frac{1}{\beta_{1i}^2} = \frac{1}{\beta_{1i1}^2} + \frac{1}{\beta_{1i2}^2}, \tag{16}$$

where $i = 1, 2$, solution of Eq. (13) can be written as

$$A_n^{(e)}(x,t) = \frac{A_n(x,t)}{\beta_e}, \tag{17}$$

where $A_n(x,t)$ is the solution of the three wave interaction equation for linear, i.e. unbranched waveguide.

4. Second harmonic generation

As it was shown in the previous section, the soliton solutions of the three wave interaction system, (7) can be obtained, when the constraints given by Eq. (12) are fulfilled. Note that, we assume that the interaction of (incoming) fundamental harmonics occurs in the first branch of the star-shaped waveguide, i, e, at $t = 0$, $A_{1,2}^{(j)}$ start to interact being located at the first bond, while, $A_3^{(j)} = 0$ at $t = 0$. The following values for the parameters of the solution are chosen: $A_{10} = A_{20} = 1$, $\upsilon_1 = 2.5$, $\upsilon_2 = 1$, $\upsilon_3 = 1.8$, $\sigma = 1$ and $T = 0$.

Fig. 3 represents plots of the profiles of the fundamental and (generated) second harmonics, i.e., solitons solutions of Eq. (7) at different time moments for the case when sum rule in Eq. (12) is fulfilled. Two important effects can be observed from these plots: i) Generation of the second harmonic occurs near the branching point; ii) there is no backscattering of fundamental harmonics at the branching point, i.e. transmission of $A_1^{(j)}$ and $A_2^{(j)}$ through the branching point is reflectionless. As it can be seen from the plots, the generation of second harmonics occurs in the second and third branches of the waveguide in this case. All the above results show that by choosing the parameters β_j appropriately, one can achieve second harmonic generation and soliton propagation in optical waveguide networks.

5. Conclusions

In this paper, we have studied the generation and dynamics of the second harmonics in optical waveguide networks in the form of solitons. The pulse generation is assumed to be caused by three-wave interactions, which are described in terms of the system of nonlinear evolution equations. We have obtained the analytic solution of the system on a metric star and tree graphs with the vertex boundary conditions, following from the fundamental conservation laws. An analytical (soliton) solution of a three-wave system of evolution equations can be obtained by fulfilling the constraint given in terms of the simple sum rule for the system parameters is shown. The results also show that for the regime, when the fundamental harmonics start to move from the first branch, the generation of second harmonics occurs at the vertex and continues in the small vicinity of the vertex during a very short time. Upon time elapses, rapid decay of the second harmonic amplitude occurs and fundamental harmonics continue to propagate along the second and third branches of the waveguide. Such dependence of second harmonic generation on the system parameters (nonlinearity parameter, β_j) allows to use it for tuning the material properties to achieve needed harmonic generation and propagation regime. The model proposed can be useful for designing and fabricating of optical structures having practical applications in short-pulse laser generations and for controlling the soliton generation in optoelectronic devices.

CRediT authorship contribution statement

M. Akramov: Visualization, Software, Investigation, Formal analysis. **B. Eshchanov:** Validation, Conceptualization. **S. Usanov:** Investigation. **Sh. Norbekov:** Visualization, Software. **D. Matrasulov:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology.

Declaration of competing interest

The authors made equal contribution into this work. There is no conflict of interests related to this paper.

Data availability

No data was used for the research described in the article.

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