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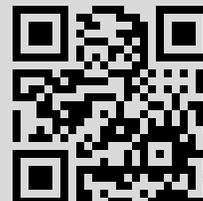
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Delta-extremal Functions in \mathbb{C}^n

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Abstract. The article is devoted to properties of a weighted Green function. We study the (δ, ψ) -extremal Green function $V_\delta^*(z, K, \psi)$ defined by the class $\mathcal{L}_\delta = \{u(z) \in psh(\mathbb{C}^n) : u(z) \leq C_u + \delta \ln^+ |z|, z \in \mathbb{C}^n\}$, $\delta > 0$. We see that the notion of regularity of points with respect to different numbers δ differ from each other. Nevertheless, we prove that if a compact set $K \subset \mathbb{C}^n$ is regular, then δ -extremal function is continuous in the whole space \mathbb{C}^n .

Keywords: plurisubharmonic function, Green function, weighted Green function, δ -extremal function.

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1. Introduction and preliminaries

The Green function in the multidimensional complex space \mathbb{C}^n is one of the main objects for the study of analytic and plurisubharmonic (*psh*) functions. The Green function was introduced and applied in the works of P. Lelong, J. Sichak, V. Zaharyuta, A. Zeriahi, A. Sadullaev and others (see [1–7]). Recall that a function $u(z) \in psh(\mathbb{C}^n)$ is said to be of logarithmic growth if there is a constant C_u such that

$$u(z) \leq C_u + \ln^+ |z|, \quad z \in \mathbb{C}^n,$$

where $\ln^+ |z| = \max\{\ln |z|, 0\}$. The family of all such functions is called the Lelong class and denoted by \mathcal{L} . We also introduce a class \mathcal{L}^+ as follows:

$$\mathcal{L}^+ := \{u(z) \in psh(\mathbb{C}^n), \quad c_u + \ln^+ |z| \leq u(z) \leq C_u + \ln^+ |z|\}.$$

For a fixed compact set $K \subset \mathbb{C}^n$ we put

$$V(z, K) = \sup\{u(z) : u(z) \in \mathcal{L}, u(z)|_K \leq 0\}.$$

Then the regularization of

$$V^*(z, K) = \overline{\lim}_{w \rightarrow z} V(w, K)$$

is called *the Green function* of the compact set K . For a non-pluripolar compact set K , the function $V^*(z, K)$ exists ($V^*(z, K) \not\equiv +\infty$) and belongs to the class \mathcal{L}^+ . The Green function $V^*(z, K) \equiv +\infty$ if and only if K is pluripolar.

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Definition 1. A compact set $K \subset \mathbb{C}^n$ is called globally pluri-regular at a point z_0 if $V^*(z_0, K) = 0$. It is called locally pluri-regular at a point z_0 if $V^*(z_0, K \cap B(z_0, r)) = 0$ for any ball $B(z_0, r)$, $r > 0$. A compact set K is globally pluri-regular if it is globally pluri-regular at every point of itself. A compact set K is locally pluri-regular if it is locally pluri-regular at every point of itself.

Theorem 1.1 (see for example, J. Siciak [4], V. Zakharyuta [3]). If a compact set K is globally pluriregular, then the function $V^*(z, K)$ is continuous in \mathbb{C}^n , and $V^*(z, K) = V(z, K)$.

2. Weighted Green functions in \mathbb{C}^n

Let $\psi(z)$ be a bounded function on a compact set $K \subset \mathbb{C}^n$. Consider the class of functions

$$\mathcal{L}(K, \psi) := \{u(z) \in \mathcal{L}, u(z)|_K \leq \psi(z)\}$$

and

$$V(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}(K, \psi)\}, z \in \mathbb{C}^n.$$

Then $V^*(z, K, \psi) = \overline{\lim}_{w \rightarrow z} V(w, K, \psi)$ is said to be a weighted Green function of K with respect to $\psi(z)$. Note that in the case $\psi(z) \equiv 0$ the function $V^*(z, K, \psi)$ coincides with the Green function $V^*(z, K)$, i.e., $V^*(z, K, 0) \equiv V^*(z, K, \psi)$. Extremal weighted Green functions are the subject of study by many authors (see [7, 10–13]). They are successfully applied in multidimensional complex analysis, in the approximation theory of functions, in multidimensional complex dynamical systems etc.

It is clear that for any compact set $K \subset \mathbb{C}^n$ we have the inequality

$$V^*(z, K) + \min_K \psi(z) \leq V^*(z, K, \psi) \leq V^*(z, K) + \max_K \psi(z). \quad (1)$$

If a function $\psi(z)$ extends to the space \mathbb{C}^n as a function from the class \mathcal{L} , i.e. if there is a function

$$\Psi \in \mathcal{L} : \Psi|_K \equiv \psi, \quad (2)$$

then it is obvious $V(z, K, \psi) \geq \Psi(z)$ and

$$V(z, K, \psi) = \psi(z) \quad \forall z \in K. \quad (3)$$

However, if the condition (2) is not met, then generally speaking, the equality (3) is not true.

Example 1. Let $K = \{|z| \leq 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$. Then by the maximum principle

$$V(z, K, \psi) = V(z, K) = V(z, K) = \ln^+ |z|.$$

Therefore, $V(z, K, \psi) = 0 < \psi(z) \quad \forall |z| < 1$.

According to this example, in order to introduce the concept of regularity, below we assume that the Green function satisfies the condition (3).

Definition 2. We say that a compact set K is globally ψ -regular at z^0 if $V^*(z^0, K, \psi) = \psi(z^0)$. We say that a compact set K is locally ψ -regular at z^0 if $V^*(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ for every ball $B(z^0, r)$, $r > 0$.

A. Sadullaev [7] proved the following theorem.

Theorem 2.1. *Let K be a compact set, and $\psi(z)$ is a weight on K such that there exists a strictly plurisubharmonic function*

$$\Psi \in \mathcal{L} \cap C^2(\mathbb{C}^n) : dd^c \Psi > 0, \Psi|_K = \psi. \quad (4)$$

Then K is locally ψ -regular at $z^0 \in K$ if and only if K is globally ψ -regular at z^0 .

Note that Theorem 2.1, generally speaking, is not true if Ψ is not a strictly plurisubharmonic function. For the weight function $\psi(z) \equiv 0$ and for the compact set $K = \{|z| = 1\} \cup \{z = 0\} \subset \mathbb{C}$ the point $z = 0$ is globally regular, but it is not locally regular. In this example K is not polynomially convex $\hat{K} \neq K$. In the work [5] A. Sadullaev constructed the following interesting example.

Example 2. The compact set $K = K_1 \cup K_2 \subset \mathbb{C}^2(z_1, z_2)$, where $K_1 = \{|z_1| < 1, z_2 = 0\}$, $K_2 = \{z_1 = e^{i\varphi}, \operatorname{Re} z_2 = 0, 0 \leq \operatorname{Im} z_2 \leq e^{\frac{1}{\cos \varphi - 1}}, -\pi \leq \varphi \leq \pi\}$, has the following properties:

- a) K is polynomially convex, i.e., $\hat{K} = K$;
- b) K is globally pluri-regular, i.e., $V^*(z, K) = 0, \forall z \in K$;
- c) K is not locally pluri-regular at the points $z \in K_1$.

In connection with this example and with Theorem 2.1, the following problem arises (see [7]).

Problem 1. *Let K be a compact set in \mathbb{C}^n . Under a weaker condition that the weight function $\psi(z)$ continues only to a neighbourhood $U \supset K$ as a strictly plurisubharmonic function, prove that K is locally ψ -regular at $z_0 \in K$ if and only if K is globally ψ -regular at $z_0 \in K$.*

The following theorem relates to local regularity for different weight functions.

Theorem 2.2. *Let K be a compact set, and $\psi(z)$ is a weight on $K : \psi(z) \in C(K)$. Then K is locally ψ -regular at $z^0 \in K$ if and only if K is locally regular (case $\psi \equiv 0$) at z^0 .*

Proof. Indeed, we use the inequality (1). If the point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \bar{B}) = \sigma > 0$ for some neighborhood $B : z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \bar{B}_1) \geq \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (1)

$$V^*(z^0, K \cap B_1, \psi) \geq V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geq \sigma + \min_{K \cap B_1} \psi(z). \quad (5)$$

Since $\psi(z)$ is continuous, choosing the neighborhood B_1 small enough we can make the right part of (5) to be greater than $\psi(z^0)$ i.e., $V^*(z, K \cap B_1, \psi) > \psi(z^0)$ and the point z^0 is not locally ψ -regular.

Reversing the roles of $V^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (1) we can prove the second part of the theorem: if the point $z^0 \in K$ is not locally ψ -regular, then it is not locally pluri-regular. \square

It should be noted here that the conditions of continuity of the function $\psi(z)$ in Theorem 2.2 is essential. An example is given in [15], when the function $\psi(z)$ is discontinuous, Theorem 2.2 is false, i.e., some point $z^0 \in K \subset \mathbb{C}$ is a ψ -regular point, but it is not pluri-regular.

3. δ -extremal functions

Let $K \subset \mathbb{C}^n$ be a compact set and $\psi(z)$ be some bounded function on K . Consider the following generalization of the Lelong class

$$\mathcal{L}_\delta := \{u(z) \in psh(\mathbb{C}^n) : u(z) \leq C_u + \delta \ln^+ |z|, z \in \mathbb{C}^n\}, \delta > 0.$$

It is clear that if $v(z) \in \mathcal{L}$, then $c \cdot v(z) \in \mathcal{L}_\delta$, where $0 < c \leq \delta$. Put

$$\mathcal{L}_\delta(K, \psi) := \{u(z) \in \mathcal{L}_\delta, u(z)|_K \leq \psi(z)\}.$$

Definition 3. The function $V_\delta^*(z, K, \psi) = \overline{\lim}_{w \rightarrow z} V_\delta(w, K, \psi)$ is called a δ -extremal function of K with respect to $\psi(z)$, where

$$V_\delta(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}_\delta(K, \psi)\}, z \in \mathbb{C}^n.$$

We list simple properties of δ -extremal functions:

- 1°. If $\delta_1 \leq \delta_2$, then $V_{\delta_1}(z, K, \psi) \leq V_{\delta_2}(z, K, \psi)$.
- 2°. If $\psi_1 \leq \psi_2, \forall z \in K$, then $V_\delta(z, K, \psi_1) \leq V_\delta(z, K, \psi_2)$.
- 3°. $V_\delta(z, K, \psi) = \delta V(z, K, \frac{\psi}{\delta})$, in particular $V_\delta(z, K) = \delta V(z, K)$.
- 4°. $V_\delta(z, K, \psi + c) = c + V_\delta(z, K, \psi), \forall c \in \mathbb{R}$.

If a function $\psi(z)$ extends to the space \mathbb{C}^n as a function from the class \mathcal{L}_δ , i.e. if there is a function

$$\Psi \in \mathcal{L}_\delta : \Psi|_K \equiv \psi, \quad (6)$$

then it is obvious $V_\delta(z, K, \psi) \geq \Psi(z)$ and

$$V_\delta(z, K, \psi) = \psi(z) \quad \forall z \in K. \quad (7)$$

However, if the condition (6) is not met, then generally speaking, the equality (7) is not true. In this section, as above we assume that the Green function $V_\delta(z, K, \psi)$ satisfies the condition (7). For such a function ψ we can introduce the concept of (δ, ψ) -regularity.

Definition 4. We say that a compact set K is globally (δ, ψ) -regular at z^0 if $V_\delta^*(z^0, K, \psi) = \psi(z^0)$. We say that a compact set K is locally (δ, ψ) -regular at z^0 if $V_\delta^*(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ for any ball $B(z^0, r), r > 0$.

The following theorem is proved similarly to the proof of Theorem 2.2 and we omit it.

Theorem 3.1. Let K be a compact set and $\psi(z)$ is a weight on $K : \psi(z) \in C(K), V_\delta(z, K, \psi) = \psi(z) \forall z \in K$. Then K is locally (δ, ψ) -regular at $z^0 \in K$ if and only if K is locally $(\delta, 0)$ -regular at z^0 .

Similarly to Theorem 1.1 the continuity of the δ -extremal function takes place.

Theorem 3.2. Let $\psi(z)$ be continuous on K . If K is globally (δ, ψ) -regular i.e. if K is globally (δ, ψ) -regular at a point $z^0 \in K$, then $V_\delta^*(z, K, \psi) = V_\delta(z, K, \psi)$ and $V_\delta^*(z, K, \psi)$ is continuous in \mathbb{C}^n .

Proof. Let $\psi(z)$ be a function defined and continuous on K . It is well known that $\psi(z)$ can be extended continuously to K , i.e., there is a function $\Psi(z) \in C(\mathbb{C}^n)$ such that $\Psi(z)|_K = \psi(z)$ (see Whitney H. [8]). We use the standard approximation $u_j \downarrow V_\delta^*(z, K, \psi)$, where $u_j \in \mathcal{L}_\delta \cap C^\infty(\mathbb{C}^n)$. Since $V_\delta^*(z, K, \psi) \equiv \Psi(z)$, $z \in K$, for any $\varepsilon > 0$ there is an open set $\{z \in \mathbb{C}^n, V_\delta^*(z, K, \psi) < \Psi(z) + \varepsilon\}$ contained K . Therefore, by the Hartogs lemma, there exists $j_0 \in \mathbb{N}$ such that $u_j(z) < \Psi(z) + 2\varepsilon = \psi(z) + 2\varepsilon$, $\forall z \in K$, $j > j_0$. From here, $u_j - 2\varepsilon \in \mathcal{L}_\delta(\psi, K)$ and

$$u_j - 2\varepsilon \leq V_\delta(z, K, \psi) \leq V_\delta^*(z, K, \psi) \leq u_j, \quad j > j_0, \quad z \in \mathbb{C}^n.$$

This means that the sequence u_j converges to $V_\delta^*(z, K, \psi)$ uniformly and $V_\delta^*(z, K, \psi) = V_\delta(z, K, \psi) \in C(\mathbb{C}^n)$. \square

In the case when $\delta = 1$ and $\psi(z)$ continues throughout \mathbb{C}^n as a continuous function of the class \mathcal{L} , Theorem 3.2 was proved by A. Sadullaev.

4. δ -extremal functions for different δ

Note that in the general case $V_\delta(z, K, \psi)$ and the weight function ψ do not have to be equal on K for all δ . In other words, the condition (7) may not be satisfied.

Example 3 (see Alan [10]). Let $K = \overline{B(0, 1)}$ and $\psi(z) = |z|^2$. Then one can prove that

$$V_\delta(z, K, \psi) = \begin{cases} |z|^2, & |z| \leq \sqrt{\frac{\delta}{2}}, \\ \delta \ln |z| + \frac{\delta}{2} - \frac{\delta}{2} \ln \left| \frac{\delta}{2} \right|, & |z| > \sqrt{\frac{\delta}{2}}. \end{cases}$$

We see $V_\delta(z, K, \psi) = |z|^2$, $\forall z \in \left\{ |z| \leq \sqrt{\frac{\delta}{2}} \right\}$ and $V_\delta(z, K, \psi) < |z|^2$, $\forall z \in \left\{ \sqrt{\frac{\delta}{2}} < |z| \leq 1 \right\}$.

We denote by $\Lambda = \Lambda(K, \psi)$ the set of numbers δ for which the equality of type (7) holds, i.e.

$$\Lambda = \Lambda(K, \psi) = \{ \delta > 0 : V_\delta(z, K, \psi)|_K \equiv \psi(z) \}.$$

For Alan's example, $\Lambda = [2, +\infty)$. In fact,

$$V_2(z, K, \psi) = \begin{cases} |z|^2, & |z| \leq 1, \\ 2 \ln |z| + 1, & |z| > 1. \end{cases}$$

So, $V_2(z, K, \psi)|_K \equiv \psi(z)$ and by property 1° from Section 3 $V_\delta(z, K, \psi) \geq V_2(z, K, \psi)$ for all $\delta \in [2, +\infty)$. If $\delta \in (0, 2)$ then there is a point $z^0 \in K$ such that $V_\delta(z^0, K, \psi) < \psi(z^0)$, that is $(0, 2) \cap \Lambda = \emptyset$.

The sets Λ may be empty. For example, for $K = \{|z| \leq 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$, by property 3° we have

$$V_\delta(z, K, \psi) = V_\delta(z, K) = \delta V(z, K) = \delta \ln^+ |z|.$$

Therefore, for any $\delta > 0$, $V_\delta(z, K, \psi) < \psi(z)$, $\forall |z| < 1$. That is, in this case $\Lambda = \emptyset$.

If $\psi(z) \equiv c$, where c is a constant, then $V_\delta(z, K, c) = c + V_\delta(z, K) = c + \delta V(z, K)$. Since the Green function $V(z, K) \geq 0$, for any $\delta > 0$ and $z \in K$ the equality $V_\delta(z, K, c) = c$ holds. This means that $\Lambda = (0, +\infty)$.

Let $\Lambda \neq \emptyset$. If $\delta \in \Lambda$, then from property 1° we easily get $\delta_1 \in \Lambda$ for $\delta_1 > \delta$. On the other hand

Proposition 1. *If $\delta_j \in \Lambda$, $\forall j \in \mathbb{N}$ and $\delta_j \downarrow \delta_0 \neq 0$ as $j \rightarrow \infty$ then $\delta_0 \in \Lambda$.*

Proof. Indeed, by the hypothesis we have $V_{\delta_j}(z, K, \psi) = \psi(z)$, $z \in K$. Using properties 2° and 3°, we get

$$V_{\delta_j}(z, K, \psi) = \delta_j V\left(z, K, \frac{\psi}{\delta_j}\right) \leq \delta_j V\left(z, K, \frac{\psi}{\delta_0}\right).$$

Consequently, $\forall j \in \mathbb{N}$ we have $\psi(z) = V_{\delta_j}(z, K, \psi) \leq \delta_j V\left(z, K, \frac{\psi}{\delta_0}\right)$, $z \in K$. As j tends to infinity, we get

$$\psi(z) \leq \delta_0 V\left(z, K, \frac{\psi}{\delta_0}\right) = V_{\delta_0}(z, K, \psi), \quad z \in K,$$

i.e. $\psi(z) = \delta_0 V\left(z, K, \frac{\psi}{\delta_0}\right) = V_{\delta_0}(z, K, \psi)$, $z \in K$ and $\delta_0 \in \Lambda$. \square

Proposition 1 follows, if $\Lambda \neq \emptyset$ then $\Lambda = (0, \infty)$ or $\Lambda = [\delta_0, +\infty)$, $\delta_0 > 0$. Note that if $\delta \in \Lambda(K, \psi)$, then $V_\delta(z, K, \psi) = \psi(z)$, $z \in K$. Therefore, by monotonicity $V_\delta(z, K \cap \bar{B}, \psi) = \psi(z)$, $z \in K \cap \bar{B}$, for any ball $B \cap K \neq \emptyset$. It follows that if $\delta \in \Lambda(K, \psi)$, then $\delta \in \Lambda(K \cap B, \psi)$.

Definition 5. *Let $\delta \in \Lambda(K)$. A compact set K is called globally (δ, ψ) -regular at a point $z^0 \in K$ if $V_\delta^*(z^0, K, \psi) = \psi(z^0)$. It is called locally (δ, ψ) -regular at a point $z^0 \in K$ if for every nonempty ball $B(z^0, r) : V_\delta^*(z^0, K \cap \bar{B}(z^0, r), \psi) = \psi(z^0)$. A compact set K is globally (δ, ψ) -regular if it is globally (δ, ψ) -regular at every point of itself. A compact K is locally (δ, ψ) -regular if it is locally (δ, ψ) -regular at every point of itself.*

Note that global or local (δ, ψ) -regularity can only be defined for $\delta \in \Lambda$. It is easy to see that any locally (δ, ψ) -regular point is globally (δ, ψ) -regular. We denote by $\Lambda_{reg} = \Lambda_{reg}(K, \psi)$ the set of numbers $\delta \in \Lambda$, for which K is globally regular, we denote by $\Lambda_{reg}^{loc} = \Lambda_{reg}^{loc}(K, \psi)$ the set of numbers $\delta \in \Lambda$, for which K is locally regular. We see, $\Lambda_{reg}^{loc} \subset \Lambda_{reg} \subset \Lambda$.

Proposition 2. *Let $\delta_1, \delta_2 \in \Lambda$ and $\delta_1 \leq \delta_2$. If a point z^0 is (δ_2, ψ) -regular, then it is (δ_1, ψ) -regular.*

The proof follows from property 1° of Section 3. For a continuous function ψ there holds

Theorem 4.1. *Let $\delta \in \Lambda$, and a function $\psi(z)$ be continuous on K . Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (δ, ψ) -regular if and only if it is locally pluri-regular.*

Proof. We show that for any compact set $K \subset \mathbb{C}^n$ the following is true:

$$\delta V^*(z, K) + \min_K \psi(z) \leq V_\delta^*(z, K, \psi) \leq \delta V^*(z, K) + \max_K \psi(z). \quad (8)$$

In fact, if $u \in \mathcal{L}_\delta(K, \psi)$, i.e., $u \in \mathcal{L}_\delta$, $u|_K \leq \psi$, then

$$u(z) - \max_K \psi(z) \in \mathcal{L}_\delta(K).$$

Therefore

$$u(z) - \max_K \psi(z) \leq V_\delta^*(z, K)$$

and

$$V_\delta^*(z, K, \psi) - \max_K \psi(z) \leq V_\delta^*(z, K) = \delta V^*(z, K), \quad \forall z \in \mathbb{C}^n.$$

Conversely, if $u \in \mathcal{L}_\delta(K)$, then $u(z) + \min_K \psi(z) \in \mathcal{L}_\delta(K, \psi)$. Therefore,

$$V_\delta^*(z, K) + \min_K \psi(z) = \delta V^*(z, K) + \min_K \psi(z) \leq V_\delta^*(z, K, \psi),$$

so that (8) holds.

Using (8) we can now prove the theorem. If a fixed point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B : z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \overline{B}_1) \geq \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (8)

$$V_\delta^*(z^0, K \cap B_1, \psi) \geq \delta V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geq \delta \sigma + \min_{K \cap B_1} \psi(z). \tag{9}$$

Since $\psi(z)$ is continuous, choosing a neighborhood B_1 small enough we can make the right part of (9) to be greater than $\psi(z^0)$ i.e., $V_\delta^*(z, K \cap B_1, \psi) > \psi(z^0)$. This means that the point z^0 is not locally (δ, ψ) -regular.

Reversing the roles of $V_\delta^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (8) we can prove the second part of the theorem: if a point $z^0 \in K$ is not locally (δ, ψ) -regular, then it is not locally pluri-regular. □

Corollary 1. *Let $\delta_1, \delta_2 \in \Lambda$ and a function $\psi(z)$ be continuous on K . Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (δ_1, ψ) -regular if and only if it is locally (δ_2, ψ) -regular.*

Proposition 3. *If $\delta_j \in \Lambda_{reg}, \forall j \in \mathbb{N}$ and $\delta_j \uparrow \delta$ as $j \rightarrow \infty$, then $\delta \in \Lambda_{reg}$.*

Proof. In fact, since $\psi(z) = V_{\delta_j}^*(z, K, \psi), z \in K$, we get

$$\psi(z) = V_{\delta_j}^*(z, K, \psi) = \delta_j V^*(z, K, \frac{\psi}{\delta_j}) \geq \delta_j V^*(z, K, \frac{\psi}{\delta}).$$

Therefore, $\forall j \in \mathbb{N}$ we have $\psi(z) \geq \delta_j V^*(z, K, \frac{\psi}{\delta}), z \in K$. As j tends to infinity, we get

$$\psi(z) \geq \delta V^*(z, K, \frac{\psi}{\delta}) = V_\delta^*(z, K, \psi), z \in K.$$

This means that $\delta \in \Lambda_{reg}$. □

Corollary 2. *If $\Lambda = [\delta_0, \infty)$, then $\Lambda_{reg} = \begin{cases} \text{or } [\delta_0, \delta_1] \\ \text{or } [\delta_0, \infty). \end{cases}$*

Corollary 3. *If $\Lambda = (0, \infty)$, then $\Lambda_{reg} = \begin{cases} \text{or } (0, \delta_1] \\ \text{or } (0, \infty). \end{cases}$*

In the paper [10] M. Alan studied the concepts of (δ, ψ) -regularity and posed the following problem

Problem 2 ([10]). *Let K be a compact set in \mathbb{C}^n , $\psi(z)$ extends to $\mathcal{L}_{\delta_1}^+$ (see (6)) and $0 < \delta_1 < \delta_2$. If K is (δ_1, ψ) -regular at $z_0 \in K$, then K is (δ_2, ψ) -regular at z_0 .*

5. The property of (δ, ψ) -regularity

Further properties of δ -extremal function are associated with pluri-thin sets.

Definition 6. *Let $E \subset \mathbb{C}^n$ and let E' be its limit point set. Then E is said to be pluri-thin at z^0 if either $z^0 \notin E'$ or $z^0 \in E'$ but there exists a neighbourhood U of z^0 and a function $u(z) \in psh(U)$ such that*

$$\overline{\lim_{\substack{z \rightarrow z^0 \\ z \in E \setminus \{z^0\}}} u(z) < u(z^0).$$

So, if the set E is not thin at the point z^0 , then for any plurisubharmonic function $u(z)$ in the neighborhood of z^0

$$\overline{\lim_{\substack{z \rightarrow z^0 \\ z \in E \setminus \{z^0\}}} u(z)} = \overline{\lim_{\substack{z \rightarrow z^0 \\ z \in E}} u(z)} = u(z^0).$$

Proposition 4 ([16]). *If $E \subset \mathbb{C}^n$ is pluri-thin at a limit point z^0 of E , then there exists a plurisubharmonic function $u \in \mathcal{L}^+$ such that*

$$\overline{\lim_{\substack{z \rightarrow z^0 \\ z \in E \setminus \{z^0\}}} u(z)} = -\infty < u(z^0).$$

Theorem 5.1. *If z^0 is a pluri-thin point of K , then z^0 is locally (δ, ψ) -irregular point of K . Here the function $\psi \in L^\infty(K)$ and $\delta \in \Lambda$.*

Proof. Let K be pluri-thin at the point $z^0 \in K$. Then, according to Proposition 4, there exists a function $u(z) \in \mathcal{L}_\delta$ such that

$$\lim_{\substack{z \rightarrow z^0 \\ z \in E \setminus \{z^0\}}} u(z) = -\infty < u(z^0).$$

Without loss of generality, we can assume $u(z^0) > 0$ and find a ball $B(z^0, r)$ such that

$$\begin{cases} u(z) \leq \inf_{z \in K} \psi(z) - \psi(z^0) \text{ for } z \in K \cap B \setminus \{z^0\}, \\ u(z^0) > 0. \end{cases}$$

Put $w(z) = u(z) + \psi(z^0)$. It is easy to see that $w(z) \in L_\delta(\psi, K \cap B \setminus \{z^0\})$, because for $z \in K \cap B \setminus \{z^0\}$

$$w(z) = u(z) + \psi(z^0) \leq \inf_{z \in K} \psi(z) - \psi(z^0) + \psi(z^0) = \inf_{z \in K} \psi(z) \leq \psi(z).$$

Consequently,

$$w(z) \leq V_\delta^*(z, K \cap B \setminus \{z^0\}, \psi) = V_\delta^*(z, K \cap B, \psi), \quad \forall z \in \mathbb{C}^n.$$

From here

$$w(z^0) \leq V_\delta^*(z^0, K \cap B, \psi).$$

On the other hand

$$w(z^0) = u(z^0) + \psi(z^0) > \psi(z^0).$$

Therefore

$$\psi(z^0) < w(z^0) \leq V_\delta^*(z^0, K \cap B, \psi).$$

Hence, the point z^0 is a locally (δ, ψ) irregular point of the compact set K . \square

Note that if $n > 1$, the necessary condition of Theorem 5.1, generally speaking, is not true.

Example 4. Let $(\delta, \psi) = (1, 0)$ and $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z| \leq 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0, |z_1| \leq 2\}$.

The compact set K is a union of the unit ball in \mathbb{C}^2 and a pluripolar set. We have

$$V(z, K) = \begin{cases} \ln^+ |z| & \text{for } z_2 \neq 0 \\ \ln^+ \left| \frac{z_1}{2} \right| & \text{for } z_2 = 0 \end{cases}$$

and

$$V^*(z, K) = \ln^+ |z|.$$

A point $(2, 0) \in K$ is an irregular point, but it is not pluri-thin.

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Дельта-экстремальная функция в пространстве \mathbb{C}^n

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Аннотация. В этой статье мы изучаем (δ, ψ) -экстремальную функцию Грина $V_\delta^*(z, K, \psi)$, которая определяется при помощи класса $\mathcal{L}_\delta = \{u(z) \in psh(\mathbb{C}^n) : u(z) \leq C_u + \delta \ln^+ |z|, z \in \mathbb{C}^n\}$, $\delta > 0$. Покажем, что понятие регулярности точек для разных δ не совпадают. Тем не менее мы доказываем, что если компакт $K \subset \mathbb{C}^n$ регулярен, то δ -экстремальная функция Грина непрерывна во всем пространстве \mathbb{C}^n .

Ключевые слова: плюрисубгармонические функции, экстремальная функция Грина, функция Грина с весом, δ -экстремальная функция.