AIP Conference Proceedings

RESEARCH ARTICLE | APRIL 03 2025

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AIP Conf. Proc. 3265, 050025 (2025)
https://doi.org/10.1063/5.0265085

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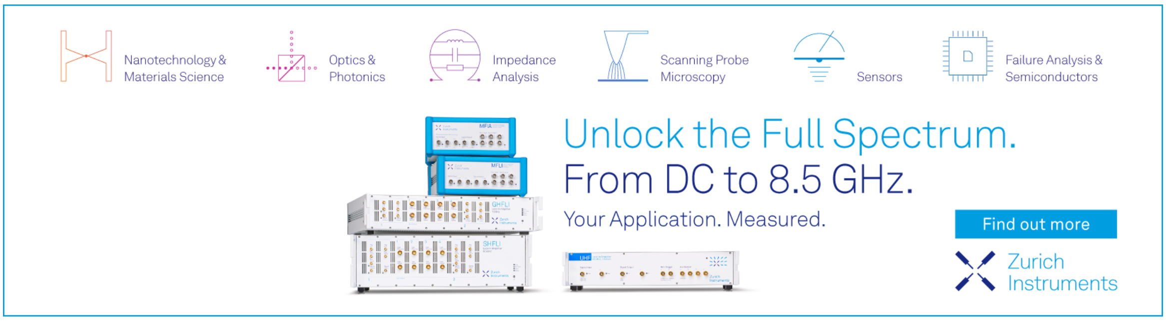

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Non-classical theory of the equilibrium of layered plates

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Abstract. An improved non-classical theory [10] on the equilibrium of layered plates is proposed in this article based on three-dimensional equations of elasticity theory without hypotheses. Here, all boundary conditions specified in the front and interlayer planes of the layered plate are met. Recurrence relations are obtained that ensure the fulfillment of interlayer continuity conditions for the components of the displacement vector and stress tensor. Equilibrium equations are derived in terms of the components of the longitudinal force tensor, internal moments, and vector of shearing forces for each "package" of layers of plates.

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INTRODUCTION

As is known, layered composites, which, along with low specific density, have high specific strength, are widely used in various fields of mechanical engineering, construction, and modern technologies for creating high-strength ultra-light aircraft [1,2]. At present, with the development of the latest technologies for high-strength materials made from composites, there is a need to study problems with a more exact solution to the equation of motion of layered plates [3-8].

In this study, without preliminary hypotheses, a new approach to constructing a non-classical theory of layered plates is proposed.

SOLUTION METHOD

The equilibrium equation of the stress state of a three-dimensional body in a Cartesian coordinate frame $Ox_1x_2x_3$, under the influence of a system of surface and volumetric forces, has the following form [8,9]:

$$\sigma_{ij},_j + X_i - \rho\ddot{U}_i = 0, \tag{1}$$

which defines relations between the stress σ_{ij} and strain ε_{ij} tensors

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

and geometric Cauchy relations between ε_{ij} and components of the displacement vector U_i

$$\varepsilon_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right) \tag{3}$$

and boundary conditions

$$U_i\big|_{\Sigma} = U_i^{\Sigma} \;\;,\;\; \sigma_{ij}n_j\big|_{\Sigma} = S_i \tag{4}$$

International Conference: "Ensuring Seismic Safety and Seismic Stability of Buildings and Structures, Applied Problems of Mechanics"
AIP Conf. Proc. 3265, 050025-1–050025-8; https://doi.org/10.1063/5.0265085
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where X_i, S_i are the volumetric and surface forces, U_i^{Σ} are specified on part of the displacement boundary, n_j is the external normal, ρ - is the density, $C_{ijkl}(x)$ - are the elastic moduli, $i, j, k, l = 1,2,3$. For isotropic materials, the following relations for the components of the elastic modulus tensor hold:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{5}$$

here: $\lambda = \dfrac{E\nu}{(1-2\nu)(1+\nu)}$, $\mu = \dfrac{E}{2(1+\nu)}$

are the Lamé coefficients, E – modulus of elasticity ν – Poisson's ratio.

Here the Cartesian coordinate frame Ox_1x_2z is located in the lower plane of the layered plate with constant thickness h. The dependence of the elastic modulus of a layered plate along the coordinate z is given in the following form [9]:

$$C_{ijkl}(x_1,x_2,z) = C^{(1)}{}_{ijkl}(x_1,x_2)H(z_1 - z) + C^{(N)}{}_{ijkl}(x_1,x_2)H(z - z_{N-1}) + \sum_{r=2}^{N-1} C^{(r)}{}_{ijkl}(x_1,x_2)H(z_r - z)H(z - z_{r-1}), \tag{6}$$

where z_r are the coordinates of the interface between the layers of the composite plate. $H(z)$ is the Heaviside function. Here, $h_r = z_{r+1} - z_r$, $r = 1,3,\cdots N-1$, $z_1 = 0$, $z_N = h$.

The sought-for solution to problem (1)-(5) for layered plates considering (6) is presented in the following form [10]:

$$\begin{cases} U_I(x_1,x_2,z) = U_I^{(1)}(x_1,x_2,z)H(z_1 - z) + U_I^{(N)}(x_1,x_2,z)H(z - z_{N-1}) + \sum\limits_{r=2}^{N-1} U_I^{(r)}(x_1,x_2,z)H(z_r - z)H(z - z_{r-1}) \\ U_Z(x_1,x_2,z) = U_Z^{(1)}(x_1,x_2,z)H(z_1 - z) + U_Z^{(N)}(x_1,x_2,z)H(z - z_{N-1}) + \sum\limits_{r=2}^{N-1} U_Z^{(r)}(x_1,x_2,z)H(z_r - z)H(z - z_{r-1}) \end{cases}. \tag{7}$$

Consequently, the components of the stress and strain tensors are also represented in a similar form, where $U_Z^{(r)}, U_I^{(r)}$, $I = 1,2$ is the sought-for solution for each "package" of layers. Here and below, capital indices take the values 1.2. Then each "package" has its corresponding system of equilibrium equations.

$$\begin{cases} \sigma_{IJ,J}^{(r)} + \sigma_{IZ,Z}^{(r)} + X_I^{(r)} = 0; \\ \sigma_{ZJ,J}^{(r)} + \sigma_{ZZ,Z}^{(r)} + X_Z^{(r)} = 0. \end{cases} \tag{8}$$

Let us introduce "fast" variable ξ for each r-"package" by the change of variables $z_r = h_r\xi$, $\xi = 0..1$. We present the sought-for solution for each r layer as a polynomial in ξ,

$$\begin{cases} U_K^{(r)} = u_K^{(r)} + \xi h_r \psi_K^{(r)} + \xi^2 h_r^2 C_K^{(r)} + \xi^3 h_r^3 D_K^{(r)}; \\ U_Z^{(r)} = w^{(r)} + \xi h_r V^{(r)} + \xi^2 h_r^2 \theta^{(r)}. \end{cases} \tag{9}$$

At the interface $z = z_r$, for each r-"package" the conditions of continuity of displacements and stress tensor are met:

$$U_I^{(r+1)} = U_I^{(r)},\ U_Z^{(r+1)} = U_Z^{(r)},\ \sigma_{IZ}^{(r+1)} = \sigma_{IZ}^{(r)},\ \sigma_{ZZ}^{(r+1)} = \sigma_{ZZ}^{(r)},\ r = 2,\dots,N-1 \tag{10}$$

where, with (2)-(5) and (9), we have the following expressions for the stress tensor components $\sigma_{IZ}^{(r)}, \sigma_{ZZ}^{(r)}$:

$$\begin{cases} \sigma_{IZ}^{(r)}(x_1,x_2,\xi) = \mu_r\left[\psi_I{}^{(r)} + 2\xi h_r C_I{}^{(r)} + 3\xi^2 h_r{}^2 D_I{}^{(r)} + \left(w^{(r)} + \xi h_r V^{(r)} + \xi^2 h_r{}^2 \theta^{(r)}\right)_{,I}\right] \\ \sigma_{ZZ}^{(r)}(x_1,x_2,\xi) = \lambda_r\left[\left(u_K{}^{(r)} + \xi h_r \psi_K{}^{(r)} + \xi^2 h_r{}^2 C_K{}^{(r)} + \xi^3 h_r{}^3 D_K{}^{(r)}\right)_{,K} + \dfrac{1-\nu_r}{\nu_r}\left(V^{(r)} + 2\xi h_r \theta^{(r)}\right)\right] \end{cases} \tag{11}$$

At the interface between the layers, considering (9)-(11), we have

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$$\begin{cases} U_I^{(r+1)}\left(x_1, x_2, 0\right) = U_I^{(r)}\left(x_1, x_2, 1\right), \\ U_Z^{(r+1)}\left(x_1, x_2, 0\right) = U_Z^{(r)}\left(x_1, x_2, 1\right), \\ \sigma_{IZ}^{(r+1)}\left(x_1, x_2, 0\right) = \sigma_{IZ}^{(r)}\left(x_1, x_2, 1\right), \\ \sigma_{ZZ}^{(r+1)}\left(x_1, x_2, 0\right) = \sigma_{ZZ}^{(r)}\left(x_1, x_2, 1\right). \end{cases} \tag{12}$$

Therefore,

$$\begin{cases} u_I^{(r+1)} = u_I^{(r)} + h_r \psi_I^{(r)} + h_r^2 C_I^{(r)} + h_r^3 D_I^{(r)} \\ w^{(r+1)} = w^{(r)} + h_r V^{(r)} + h_r^2 \theta^{(r)} \\ \mu_{r+1} \psi_I^{(r+1)} = \mu_r \left(\psi_I^{(r)} + 2h_r C_I^{(r)} + 3h_r^2 D_I^{(r)}\right) + \left(\mu_r - \mu_{r+1}\right) w,_I^{(r+1)}, \\ \lambda_{r+1} \dfrac{1-\nu_{r+1}}{\nu_{r+1}} V^{(r+1)} = \left(\lambda_r - \lambda_{r+1}\right) u_{I,I}^{(r+1)} + \lambda_r \dfrac{1-\nu_r}{\nu_r}\left(V^{(r)} + 2h_r \theta^{(r)}\right) \end{cases} \tag{13}$$

hence, after some calculations, we have the following recurrence relations:

$$\begin{cases} u_I^{(r)} = u_I^{(1)} + \sum\limits_{l=1}^{r-1}\left(h_l \psi_I^{(l)} + h_l^2 C_I^{(l)} + h_l^3 D_I^{(l)}\right) \\ w^{(r)} = w^{(1)} + \sum\limits_{l=1}^{r-1}\left(h_l V^{(l)} + h_l^2 \theta^{(l)}\right) \\ \psi_I^{(r)} = \psi_I^{(1)} + \dfrac{1}{\mu_r}\sum\limits_{l=1}^{r-1}\left[\mu_l\left(2h_l C_I^{(l)} + 3h_l^2 D_I^{(l)}\right) + \left(\mu_l - \mu_r\right)\left(h_l V^{(l)} + h_l^2 \theta^{(l)}\right)_{,I}\right] \\ V^{(r)} = \dfrac{\nu_{r+1}}{1-\nu_{r+1}}\left[\dfrac{1-\nu_1}{\nu_1} V^{(1)} + \dfrac{1}{\lambda_r}\sum\limits_{l=1}^{r-1}\left(\left(\lambda_l - \lambda_r\right)\left(h_l \psi_I^{(l)} + h_l^2 C_I^{(l)} + h_l^3 D_I^{(l)}\right)_{,I} + \dfrac{2\lambda_l\left(1-\nu_l\right)}{\nu_l} h_l \theta^{(l)}\right)\right] \end{cases} \quad 14)$$

In this study, we assume that only the normal distributed load $q(x_1, x_2)$ acts on the upper plane of the layered plate, and there are no body forces.

In the problem under consideration, the boundary conditions on the lower plane of the layered plate, i.e., in the first layer $(\xi = 0)$, have the following form for stresses:

$$\begin{cases} \sigma_{IZ}^{(1)} = \mu_1\left[\psi_I^{(1)} + w,_I^{(1)}\right] \\ \sigma_{ZZ}^{(1)} = \lambda_1\left[u_{K,K}^{(1)} + \dfrac{1-\nu_1}{\nu_1} V^{(1)}\right] \end{cases} \tag{15}$$

Hence, we obtain:

$$\begin{cases} w,_I^{(1)} = -\psi_I^{(1)} \\ u_{K,K}^{(1)} = -\dfrac{1-\nu_1}{\nu_1} V^{(1)} \end{cases} \tag{16}$$

On the upper plane, to which the last layer corresponds, the boundary conditions for $\xi = 1$ have the following form:

$$\sigma_{IZ}^{(N)} = 0, \ \sigma_{ZZ}^{(N)} = q, \tag{17}$$

hence, considering recurrent formulas (13) and relation (16), we obtain the following system of differential equations:

$$\begin{cases} \psi_I^{(N)} + 2h_N C_I^{(N)} + 3h_N^2 D_I^{(N)} + \left(w^{(N)} + h_N V^{(N)} + h_N^2 \theta^{(N)}\right)_{,I} = 0 \\ \left(u_K^{(N)} + h_N \psi_K^{(N)} + h_N^2 C_K^{(N)} + h_N^3 D_K^{(N)}\right)_{,K} + \dfrac{1-\nu_N}{\nu_N}\left(V^{(N)} + 2h_N \theta^{(N)}\right) = \dfrac{q}{\lambda_N} \end{cases} \tag{18}$$

From (16) and (18), we can obtain the following expressions for the unknown functions of the first layer:

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$$\begin{cases} \mu_1 h_1 V,_I^{(1)} = -\sum_{r=2}^{N} \mu_r h_r V,_I^{(r)} - \sum_{r=1}^{N} \mu_r \left(2h_r C_I^{(r)} + 3h_r^2 D_I^{(r)} + h_r^2 \theta,_I^{(r)}\right) \\ \lambda_1 h_1 \psi_{K,K}^{(1)} = q - \lambda_r \sum_{r=1}^{N} \left[\left(h_r^2 C_K^{(r)} + h_r^3 D_K^{(r)}\right)_{,K} + \frac{2(1-\nu_r)}{\nu_r} h_r \theta^{(r)} \right] - \sum_{r=2}^{N} \lambda_r h_r \psi_{K,K}^{(r)} \\ w,_I^{(1)} = -\psi_I^{(1)}, \quad u_{K,K}^{(1)} = -\frac{1-\nu_1}{\nu_1} V^{(1)} \end{cases} \tag{19}$$

Thus, we have expressions for the unknown functions of the first layer of the plate under consideration.

In the absence of body forces, we transfer the equilibrium equations relative to the Ox_1x_2z coordinate frame into the $Ox_1x_2\xi$ coordinate frame:

$$\sigma_{IJ,J}^{(r)} + \frac{\sigma_{IZ,\xi}^{(r)}}{h_r} = 0$$
$$\sigma_{ZJ,J}^{(r)} + \frac{\sigma_{ZZ,\xi}^{(r)}}{h_r} = 0 \tag{20}$$

In this case, for the longitudinal components of the stress tensor $\sigma_{IJ}^{(r)}$, we have the following expressions:

$$\sigma_{IJ}^{(r)}\left(x_1, x_2, \xi\right) = \lambda_r \left[u_{K,K}^{(r)} + \xi h_r \psi_{K,K}^{(r)} + \xi^2 h_r^2 C_{K,K}^{(r)} + \xi^3 h_r^3 D_{K,K}^{(r)} + V^{(r)} + 2\xi h_r \theta^{(r)} \right] \delta_{IJ} +$$
$$+ \mu_r \left[\left(u_{I,J}^{(r)} + u_{J,I}^{(r)}\right) + \xi h_r \left(\psi_{I,J}^{(r)} + \psi_{J,I}^{(r)}\right) + \xi^2 h_r^2 \left(C_{I,J}^{(r)} + C_{J,I}^{(r)}\right) + \xi^3 h_r^3 \left(D_{I,J}^{(r)} + D_{J,I}^{(r)}\right) \right] \tag{21}$$

Using expressions (11) and (21), we calculate the longitudinal forces $N_{IJ}^{(r)} = h_r \int_0^1 \sigma_{IJ}^{(r)} d\xi$, internal moments $M_{IJ}^{(r)} = h_r^2 \int_0^1 \sigma_{IJ}^{(r)} \xi d\xi$, and shearing forces $Q_I^{(r)} = h_r \int_0^1 \sigma_{ZI}^{(r)} d\xi$:

$$N_{IJ}^{(r)} = \lambda_r h_r \left[u_{K,K}^{(r)} + \frac{1}{2} h_r \psi_{K,K}^{(r)} + \frac{1}{3} h_r^2 C_{K,K}^{(r)} + \frac{1}{4} h_r^3 D_{K,K}^{(r)} + V^{(r)} + h_r \theta^{(r)} \right] \delta_{IJ} +$$
$$+ \mu_r h_r \left[\left(u_{I,J}^{(r)} + u_{J,I}^{(r)}\right) + \frac{1}{2} h_r \left(\psi_{I,J}^{(r)} + \psi_{J,I}^{(r)}\right) + \frac{1}{3} h_r^2 \left(C_{I,J}^{(r)} + C_{J,I}^{(r)}\right) + \frac{1}{4} h_r^3 \left(D_{I,J}^{(r)} + D_{J,I}^{(r)}\right) \right] \tag{22}$$

$$M_{IJ}^{(r)} = \lambda_r h_r^2 \left[\frac{1}{2} u_{K,K}^{(r)} + \frac{1}{3} h_r \psi_{K,K}^{(r)} + \frac{1}{4} h_r^2 C_{K,K}^{(r)} + \frac{1}{5} h_r^3 D_{K,K}^{(r)} + \frac{1}{2} V^{(r)} + \frac{2}{3} h_r \theta^{(r)} \right] \delta_{IJ} +$$
$$+ \mu_r h_r^2 \left[\frac{1}{2}\left(u_{I,J}^{(r)} + u_{J,I}^{(r)}\right) + \frac{1}{3} h_r \left(\psi_{I,J}^{(r)} + \psi_{J,I}^{(r)}\right) + \frac{1}{4} h_r^2 \left(C_{I,J}^{(r)} + C_{J,I}^{(r)}\right) + \frac{1}{5} h_r^3 \left(D_{I,J}^{(r)} + D_{J,I}^{(r)}\right) \right] \tag{23}$$

$$Q_I^{(r)} = \mu_r h_r \left[\psi_I^{(r)} + h_r C_I^{(r)} + h_r^2 D_I^{(r)} + \left(w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r^2 \theta^{(r)} \right)_{,I} \right] \tag{24}$$

To derive the equilibrium equation in terms of longitudinal forces, internal moments, and shear forces, considering boundary conditions (10) and expressions for normal stresses (21) in the equilibrium equations (20), we perform the standard integration procedure over the thickness of each layer:

$$\begin{cases} N_{IJ,J}^{(r)} + \sigma_{IZ}^{(r+1)}\Big|_{\xi=0} - \sigma_{IZ}^{(r)}\Big|_{\xi=0} = 0 \\ M_{IJ,J}^{(r)} - Q_I^{(r)} + h_r \sigma_{IZ}^{(r+1)}\Big|_{\xi=0} = 0 \\ Q_{I,I}^{(r)} + \sigma_{ZZ}^{(r+1)}\Big|_{\xi=0} - \sigma_{ZZ}^{(r)}\Big|_{\xi=0} = 0 \end{cases} \tag{25}$$

Substituting (22)-(24) into (25) and considering (11), we obtain

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$$
\begin{cases}
\lambda_r h_r\left(u_{K,K}{}^{(r)}+\dfrac{1}{2}h_r\psi_{K,K}{}^{(r)}+\dfrac{1}{3}h_r{}^2C_{K,K}{}^{(r)}+\dfrac{1}{4}h_r{}^3D_{K,K}{}^{(r)}+V^{(r)}+h_r\theta^{(r)}\right)_{,I}+\\
\mu_r h_r\left(\left(u_{I,J}{}^{(r)}+u_{J,I}{}^{(r)}\right)+\dfrac{1}{2}h_r\left(\psi_{I,J}{}^{(r)}+\psi_{J,I}{}^{(r)}\right)+\dfrac{1}{3}h_r{}^2\left(C_{I,J}{}^{(r)}+C_{J,I}{}^{(r)}\right)+\dfrac{1}{4}h_r{}^3\left(D_{I,J}{}^{(r)}+D_{J,I}{}^{(r)}\right)\right)_{,J}+\\
\mu_{r+1}\left[\psi_I{}^{(r+1)}+w,_I{}^{(r+1)}\right]-\mu_r\left[\psi_I{}^{(r)}+w,_I{}^{(r)}\right]=0\\
\lambda_r h_r{}^2\left(\dfrac{1}{2}u_{K,K}{}^{(r)}+\dfrac{1}{3}h_r\psi_{K,K}{}^{(r)}+\dfrac{1}{4}h_r{}^2C_{K,K}{}^{(r)}+\dfrac{1}{5}h_r{}^3D_{K,K}{}^{(r)}+\dfrac{1}{2}V^{(r)}+\dfrac{2}{3}h_r\theta^{(r)}\right)_{,I}+\\
\mu_r h_r{}^2\left(\dfrac{1}{2}\left(u_{I,J}{}^{(r)}+u_{J,I}{}^{(r)}\right)+\dfrac{1}{3}h_r\left(\psi_{I,J}{}^{(r)}+\psi_{J,I}{}^{(r)}\right)+\dfrac{1}{4}h_r{}^2\left(C_{I,J}{}^{(r)}+C_{J,I}{}^{(r)}\right)+\dfrac{1}{5}h_r{}^3\left(D_{I,J}{}^{(r)}+D_{J,I}{}^{(r)}\right)\right)_{,J}-\\
\mu_r h_r\left[\psi_I{}^{(r)}+h_rC_I{}^{(r)}+h_r{}^2D_I{}^{(r)}+\left(w^{(r)}+\dfrac{1}{2}h_rV^{(r)}+\dfrac{1}{3}h_r{}^2\theta^{(r)}\right)_{,I}\right]+h_r\mu_{r+1}\left[\psi_I{}^{(r+1)}+w,_I{}^{(r+1)}\right]=0\\
\mu_r h_r\left(\psi_I{}^{(r)}+h_rC_I{}^{(r)}+h_r{}^2D_I{}^{(r)}+\left(w^{(r)}+\dfrac{1}{2}h_rV^{(r)}+\dfrac{1}{3}h_r{}^2\theta^{(r)}\right)_{,I}\right)_{,I}+\\
\lambda_{r+1}\left[u_{K,K}{}^{(r+1)}+\dfrac{1-\nu_{r+1}}{\nu_{r+1}}V^{(r+1)}\right]-\lambda_r\left[u_{K,K}{}^{(r)}+\dfrac{1-\nu_r}{\nu_r}V^{(r)}\right]=0
\end{cases}
\qquad (26)
$$

Thus, we have a system of equations (26), boundary conditions on the upper plane (18), recurrent relations (14) on the interface of layered plates for N layers, and boundary conditions on the lower plane of the plate under consideration (15) with a total number of 11 N relative to unknowns $u_I{}^{(r)}, \psi_I{}^{(r)}, C_I{}^{(r)}, D_I{}^{(r)}, w^{(r)}, V^{(r)}, \theta^{(r)}$, i.e. a closed system of partial differential equations.

At the ends of the layered plate, there are generally three types of fixing, which in terms of stresses and displacements can be written as [10]:

$$
\begin{cases}
I. \quad \sigma_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ U_Z{}^{(r)}\big|_\Sigma = 0\\
II. \quad U_I{}^{(r)}\big|_\Sigma = 0,\ U_Z{}^{(r)}\big|_\Sigma = 0\\
III. \quad \sigma_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ \sigma_{IZ}{}^{(r)} n_I\big|_\Sigma = 0
\end{cases}
\qquad (27)
$$

In terms of integral quantities, the given boundary conditions are, accordingly, written in the following form:

$$
\begin{cases}
I. \quad N_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ M_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ w^{(r)}\big|_\Sigma = 0,\ V^{(r)}\big|_\Sigma = 0,\ \theta^{(r)}\big|_\Sigma = 0\\
II. \quad u_I^{(r)}\big|_\Sigma = 0,\ \psi_I^{(r)}\big|_\Sigma = 0,\ w^{(r)}\big|_\Sigma = 0,\ V^{(r)}\big|_\Sigma = 0,\ \theta^{(r)}\big|_\Sigma = 0\\
III. \quad N_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ M_{IJ}{}^{(r)} n_J\big|_\Sigma = 0,\ Q_I{}^{(r)} n_I\big|_\Sigma = 0,\ V,_I{}^{(r)} n_I\big|_\Sigma = 0,\ \theta,_I{}^{(r)} n_I\big|_\Sigma = 0
\end{cases}
\qquad (28)
$$

In the boundary conditions obtained for the first two types, there is no normal displacement at the edges of the plate, therefore, V, θ are trivial; for the third type in (27), it is assumed that there is no gradient for compression along the free edges of the plate. It should be noted that here typical boundary conditions are formulated. The remaining types of boundary conditions can be either obtained by a linear combination of the above conditions or formulated in each case separately.

DISCUSSION OF RESULTS

In the problem under consideration, assuming longitudinal displacements to be potentials $U_I{}^{(r)} = U,_I{}^{(r)}$, we obtain a more simplified problem:

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$$
\begin{cases}
\left(\lambda_r + 2\mu_r\right) h_r \Delta\left(u^{(r)} + \frac{1}{2} h_r \psi^{(r)} + \frac{1}{3} h_r{}^2 C^{(r)} + \frac{1}{4} h_r{}^3 D^{(r)}\right) + \lambda_r h_r \left(V^{(r)} + h_r \theta^{(r)}\right) + \mu_{r+1}\left[\psi^{(r+1)} + w^{(r+1)}\right] - \mu_r \left[\psi^{(r)} + w^{(r)}\right] = 0 \\
\left(\lambda_r + 2\mu_r\right) h_r{}^2 \Delta\left(\frac{1}{2} u^{(r)} + \frac{1}{3} h_r \psi^{(r)} + \frac{1}{4} h_r{}^2 C^{(r)} + \frac{1}{5} h_r{}^3 D^{(r)}\right) + \lambda_r h_r{}^2 \left(\frac{1}{2} V^{(r)} + \frac{2}{3} h_r \theta^{(r)}\right) - \\
-\mu_r h_r \left[\psi_I{}^{(r)} + h_r C^{(r)} + h_r{}^2 D^{(r)} + w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r{}^2 \theta^{(r)}\right] + \mu_{r+1} h_r \left[\psi^{(r+1)} + w^{(r+1)}\right] = 0 \\
\mu_r h_r \Delta\left(\psi^{(r)} + h_r C^{(r)} + h_r{}^2 D^{(r)} + w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r{}^2 \theta^{(r)}\right) + \lambda_{r+1}\left[\Delta u^{(r+1)} + \frac{1-\nu_{r+1}}{\nu_{r+1}} V^{(r+1)}\right] - \lambda_r \left[\Delta u^{(r)} + \frac{1-\nu_r}{\nu_r} V^{(r)}\right] = 0
\end{cases}
\tag{29}
$$

System (13) is reduced to the following form considering the potentials:

$$
\begin{cases}
u^{(r+1)} = u^{(r)} + h_r \psi^{(r)} + h_r{}^2 C^{(r)} + h_r{}^3 D^{(r)} \\
w^{(r+1)} = w^{(r)} + h_r V^{(r)} + h_r{}^2 \theta^{(r)} \\
\mu_{r+1} \psi^{(r+1)} = \mu_r \left(\psi^{(r)} + 2 h_r C^{(r)} + 3 h_r{}^2 D^{(r)}\right) + \left(\mu_r - \mu_{r+1}\right) w^{(r+1)} \\
\lambda_{r+1} \frac{1-\nu_{r+1}}{\nu_{r+1}} V^{(r+1)} = \left(\lambda_r - \lambda_{r+1}\right) \Delta u^{(r+1)} + \lambda_r \frac{1-\nu_r}{\nu_r} \left(V^{(r)} + 2 h_r \theta^{(r)}\right)
\end{cases}
\tag{30}
$$

Then expression (29) has the following form:

$$
\begin{cases}
\left(\lambda_r + 2\mu_r\right) h_r \Delta\left(u^{(r)} + \frac{1}{2} h_r \psi^{(r)} + \frac{1}{3} h_r{}^2 C^{(r)} + \frac{1}{4} h_r{}^3 D^{(r)}\right) + \mu_r h_r \left(2 C^{(r)} + 3 h_r D^{(r)}\right) + \left(\lambda_r + \mu_r\right) h_r \left(V^{(r)} + h_r \theta^{(r)}\right) = 0 \\
\left(\lambda_r + 2\mu_r\right) h_r{}^2 \Delta\left(\frac{1}{2} u^{(r)} + \frac{1}{3} h_r \psi^{(r)} + \frac{1}{4} h_r{}^2 C^{(r)} + \frac{1}{5} h_r{}^3 D^{(r)}\right) + \mu_r h_r{}^2 \left[C^{(r)} + 2 h_r D^{(r)}\right] + \left(\lambda_r + \mu_r\right) h_r{}^2 \left(\frac{1}{2} V^{(r)} + \frac{2}{3} h_r \theta^{(r)}\right) = 0 \\
\left(\lambda_r + \mu_r\right) \Delta\left(\psi^{(r)} + h_r C^{(r)} + h_r{}^2 D^{(r)}\right) + \mu_r \Delta\left(w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r{}^2 \theta^{(r)}\right) + 2\lambda_r \frac{1-\nu_r}{\nu_r} \theta^{(r)} = 0
\end{cases}
\tag{31}
$$

We multiply the first line by $\frac{1}{\left(\lambda_r + 2\mu_r\right) h_r}$, the second line by $\frac{1}{\left(\lambda_r + 2\mu_r\right) h_r{}^2}$, and the third line by $\frac{1}{\left(\lambda_r + \mu_r\right)}$ and obtain the following system:

$$
\begin{cases}
\Delta\left(u^{(r)} + \frac{1}{2} h_r \psi^{(r)} + \frac{1}{3} h_r{}^2 C^{(r)} + \frac{1}{4} h_r{}^3 D^{(r)}\right) + \frac{1}{2\left(1-\nu_r\right)}\left[\left(1 - 2\nu_r\right)\left(2 C^{(r)} + 3 h_r D^{(r)}\right) + V^{(r)} + h_r \theta^{(r)}\right] = 0 \\
\Delta\left(\frac{1}{2} u^{(r)} + \frac{1}{3} h_r \psi^{(r)} + \frac{1}{4} h_r{}^2 C^{(r)} + \frac{1}{5} h_r{}^3 D^{(r)}\right) + \frac{1}{2\left(1-\nu_r\right)}\left[\left(1 - 2\nu_r\right)\left(C^{(r)} + 2 h_r D^{(r)}\right) + \frac{1}{2} V^{(r)} + \frac{2}{3} h_r \theta^{(r)}\right] = 0 \\
\Delta\left(\psi^{(r)} + h_r C^{(r)} + h_r{}^2 D^{(r)}\right) + \left(1 - 2\nu_r\right) \Delta\left(w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r{}^2 \theta^{(r)}\right) + 4\left(1 - \nu_r\right) \theta^{(r)} = 0
\end{cases}
\tag{32}
$$

Thus, we have a system of three partial differential equations (32) and a system of recurrence relations (30) to determine unknowns $u^{(r)}$, $\psi^{(r)}$, $C^{(r)}$, $D^{(r)}$, $w^{(r)}$, $V^{(r)}$, $\theta^{(r)}$. Here, for the sought-for variables, the approximate analytical Navier method can be used, ensuring the selection of basic functions that satisfy the corresponding boundary conditions. Without specifying the boundary conditions and solving the corresponding spectral problem, in (32) instead of the Laplace operator we introduce a formal substitution $-\gamma^2$ and system (32) acquires the following form:

$$
\begin{cases}
-2\left(1-\nu_r\right)\gamma^2 \left(u^{(r)} + \frac{1}{2} h_r \psi^{(r)} + \frac{1}{3} h_r{}^2 C^{(r)} + \frac{1}{4} h_r{}^3 D^{(r)}\right) + \left(1 - 2\nu_r\right)\left(2 C^{(r)} + 3 h_r D^{(r)}\right) + V^{(r)} + h_r \theta^{(r)} = 0 \\
-2\left(1-\nu_r\right)\gamma^2 \left(\frac{1}{2} u^{(r)} + \frac{1}{3} h_r \psi^{(r)} + \frac{1}{4} h_r{}^2 C^{(r)} + \frac{1}{5} h_r{}^3 D^{(r)}\right) + \left(1 - 2\nu_r\right)\left(C^{(r)} + 2 h_r D^{(r)}\right) + \frac{1}{2} V^{(r)} + \frac{2}{3} h_r \theta^{(r)} = 0 \\
-\gamma^2 \left(\psi^{(r)} + h_r C^{(r)} + h_r{}^2 D^{(r)}\right) - \left(1 - 2\nu_r\right)\gamma^2 \left(w^{(r)} + \frac{1}{2} h_r V^{(r)} + \frac{1}{3} h_r{}^2 \theta^{(r)}\right) + 4\left(1 - \nu_r\right) \theta^{(r)} = 0
\end{cases}
\tag{33}
$$

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Then from (33), we can determine $C^{(r)},\ D^{(r)},\ \theta^{(r)}$

$$\begin{cases} C^{(r)} = \dfrac{1}{(1-2\nu_r)}\left((1-\nu_r)\gamma^2 u^{(r)} - \dfrac{1}{2}V^{(r)}\right) \\ D^{(r)} = -\dfrac{\gamma^2}{12(1-\nu_r)(1-2\nu_r)}\left(\psi^{(r)} + w^{(r)}\right) \\ \theta^{(r)} = \dfrac{\gamma^2}{4}\left(\dfrac{1}{(1-\nu_r)}\psi^{(r)} + \dfrac{(1-2\nu_r)}{(1-\nu_r)}w^{(r)} + \dfrac{2}{(1-2\nu_r)}h_r V^{(r)}\right) \end{cases} \tag{34}$$

Considering the introduced potentials for the sought-for functions, substituting the expressions from system (19) into system (14) using (34) we obtain the following system

$$\begin{cases} u^{(1)} = \dfrac{(1-\nu_1)(1-2\nu_1)^2}{\mu_1\gamma^2 h_1^2\nu_1^2(3-2\nu_1)}\left\{\dfrac{q}{\gamma^2} + \sum\limits_{k=2}^{N}\dfrac{\mu_k h_k \nu_k}{(1-2\nu_k)^2}\left[2\gamma^2 h_k(1-\nu_k)u^{(k)} - \left(\dfrac{(1-2\nu_k)^2}{\nu_k} + \dfrac{\gamma^2 h_k^2}{6(1-\nu_k)}\right)\left(\psi^{(k)} + w^{(k)}\right) - \dfrac{h_k(2-\nu_k)}{\nu_k}V^{(k)}\right]\right\} \\ V^{(1)} = \dfrac{(1-2\nu_1)^2}{\mu_1 h_1^2\nu_1(3-2\nu_1)}\left\{\dfrac{q}{\gamma^2} + \sum\limits_{k=2}^{N}\dfrac{\mu_k h_k \nu_k}{(1-2\nu_k)^2}\left[2\gamma^2 h_k(1-\nu_k)u^{(k)} - \left(\dfrac{(1-2\nu_k)^2}{\nu_k} + \dfrac{\gamma^2 h_k^2}{6(1-\nu_k)}\right)\left(\psi^{(k)} + w^{(k)}\right) - \dfrac{h_k(2-\nu_k)}{\nu_k}V^{(k)}\right]\right\} \\ \psi^{(1)} = -\dfrac{2(1-\nu_1)}{\mu_1\gamma^2 h_1^2\nu_1}\left\{\mu_1 h_1\left(\dfrac{2}{\nu_1} + \dfrac{\gamma^2 h_1^2}{2(1-2\nu_1)}\right)V^{(1)} + \sum\limits_{k=2}^{N}\dfrac{\mu_k h_k}{(1-2\nu_k)}\left[2\gamma^2(1-\nu_k)u^{(k)} - \dfrac{\gamma^2 h_k\nu_k}{2(1-\nu_k)}\psi^{(k)} - \gamma^2 h_k\nu_k w^{(k)} + \left(\dfrac{\gamma^2 h_k^2}{2} - 2\nu_k\right)V^{(k)}\right]\right\} \\ w^{(1)} = \dfrac{2(1-\nu_1)}{\mu_1\gamma^2 h_1^2\nu_1}\left\{\mu_1 h_1\left(\dfrac{2}{\nu_1} + \dfrac{\gamma^2 h_1^2}{2(1-2\nu_1)}\right)V^{(1)} + \sum\limits_{k=2}^{N}\dfrac{\mu_k h_k}{(1-2\nu_k)}\left[2\gamma^2(1-\nu_k)u^{(k)} - \dfrac{\gamma^2 h_k\nu_k}{2(1-\nu_k)}\psi^{(k)} - \gamma^2 h_k\nu_k w^{(k)} + \left(\dfrac{\gamma^2 h_k^2}{2} - 2\nu_k\right)V^{(k)}\right]\right\} \end{cases} \tag{35}$$

From (31) after some calculations, we will have the following recurrence relations

$$\begin{cases} u^{(r)} = u^{(1)} + \sum\limits_{k=1}^{r-1}h_k\left(\dfrac{\gamma^2 h_k(1-\nu_k)}{(1-2\nu_k)}u^{(k)} + \left(1 - \dfrac{\gamma^2 h_k^2}{12(1-\nu_k)(1-2\nu_k)}\right)\psi^{(k)} - \dfrac{\gamma^2 h_k^2}{12(1-\nu_k)(1-2\nu_k)}w^{(k)} - \dfrac{h_k}{2(1-2\nu_k)}V^{(k)}\right) \\ w^{(r)} = w^{(1)} + \sum\limits_{k=1}^{r-1}h_k\left(\dfrac{\gamma^2 h_k}{4(1-\nu_k)}\psi^{(k)} + \dfrac{\gamma^2 h_k(1-2\nu_k)}{4(1-\nu_k)}w^{(k)} + \left(1 + \dfrac{\gamma^2 h_k^2}{2(1-2\nu_k)}\right)V^{(k)}\right) \\ \psi^{(r)} = \psi^{(1)} + \sum\limits_{k=1}^{r-1}h_k\left[\dfrac{\mu_k}{\mu_r(1-2\nu_k)}\left(2\gamma^2(1-\nu_k)u^{(k)} - \dfrac{\gamma^2 h_k\nu_k}{2(1-\nu_k)}\psi^{(k)} - \gamma^2 h_k\nu_k w^{(k)} + \left(\dfrac{\gamma^2 h_k^2}{2} - 2\nu_k\right)V^{(k)}\right) - \right. \\ \left. - \dfrac{\gamma^2 h_k}{4(1-\nu_k)}\psi^{(k)} - \dfrac{\gamma^2 h_k(1-2\nu_k)}{4(1-\nu_k)}w^{(k)} - \left(1 + \dfrac{\gamma^2 h_k^2}{2(1-2\nu_k)}\right)V^{(k)}\right] \\ V^{(r)} = \dfrac{\nu_r}{1-\nu_r}\left\{\dfrac{1-\nu_1}{\nu_1}V^{(1)} - \sum\limits_{k=1}^{r-1}\gamma^2 h_k\left[\left(\dfrac{\lambda_k}{\lambda_r} - 1\right)\left(\dfrac{\gamma^2 h_k(1-\nu_k)}{(1-2\nu_k)}u^{(k)} + \left(1 - \dfrac{\gamma^2 h_k^2}{12(1-\nu_k)(1-2\nu_k)}\right)\psi^{(k)} - \right.\right.\right. \\ \left.\left.\left. - \dfrac{\gamma^2 h_k^2}{12(1-\nu_k)(1-2\nu_k)}w^{(k)} - \dfrac{h_k}{2(1-2\nu_k)}V^{(k)}\right) - \dfrac{\lambda_k}{\lambda_r}\left(\dfrac{1}{2\nu_k}\psi^{(k)} + \dfrac{(1-2\nu_k)}{2\nu_k}w^{(k)} + \dfrac{h_k(1-\nu_k)}{\nu_k(1-2\nu_k)}V^{(k)}\right)\right]\right\} \end{cases} \tag{36}$$

and (35) substituting into (36), we will ultimately have a system of 4 N algebraic equations for unknowns $u^{(r)}, \psi^{(r)}$, $w^{(r)}, V^{(r)}$. It should be especially noted here that it is possible to study the stress-strain state of plates with an arbitrary number of layers. This fact is especially important when studying layered structural elements using advances in nanotechnology.[11]

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CONCLUSION

A non-classical theory of layered plates was proposed, ensuring interlayer continuity conditions without simplifying assumptions and hypotheses. When solving applied problems, there are no restrictions on the number of layers and interlayer connection conditions. In future studies, each "package" of layers can be anisotropic and thermoplastic.

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