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Problems of the Theory of Elasticity in Stresses

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Abstract. Unfortunately, the classical formulation of the problem of the theory of elasticity in stresses consists in solving three equilibrium equations when three static boundary conditions are meet. In this paper, based on the correct formulation of the stress problem for six Beltrami-Mitchell equations when three boundary-line equilibrium equations and three static boundary conditions are satisfied, solutions of stress problems in an approximate analytical form are obtaining. The limiting cases of a parallelepiped in the form of a rod and a plate are considered.
Keywords: Stress problem, Beltrami-Mitchell equation, correction formulation.

INTRODUCTION

As is known, in engineering applications, components of the stress tensor are using to determine the strength and stiffness of structural elements, whereas the classical formulation of the problem of elasticity theory is given in displacements. At the same time, to determine the components of the stress tensor, the vast majority of applied methods resort to numerical differentiation of the components of the displacement vector, as a result of which there is a sharp decrease in the accuracy of the obtained numerical results. Moreover, depending on the methods used, problems arise in satisfying the boundary conditions in stresses.

It follows from the literature sources that the first attempt to formulate the problem of the theory of elasticity in stresses was madeы in the works of A. N. Konovalov [1-8]. In progress Pobedri B. E. [9-12] proposed a new correct problem of the dynamic theory of elasticity in stresses. In the work of T. Kholmatov [13-15], a variational-difference method for solving the problem in stresses is proposed. In [16-17], the proposed method is used to solve problems of quasi-static equilibrium of a viscoelastic parallelepiped under stresses under the influence of mutually balanced loads, including concentrated forces.

In this paper, an approximate analytical approach to solving the problem in stresses is proposed for mutually balanced loads.

PROBLEM STATEMENT

The classical formulation of the problem of the theory of elasticity in stresses consists in solving three equilibrium equations

$$\sigma_{ij,j} + X_i = 0, \qquad x \in V, \tag{1}$$

when static boundary conditions are met

$$\sigma_{ij} n_j \big|_G = S_i, \qquad x \in G \tag{2}$$

where σ_{ij} are the kcomponents of the symmetric stress tensor. X_i and S_i are the components of mass and surface forces, respectively. n_j −components of the external normal vector. At the same time, there is, a lack of resolving equations and boundary conditions for the correct formulation of the problem in stresses.

Using the physical equations of state between the components of stress tensors $-\sigma_{ij}$ and strain tensors $-\varepsilon_{ij}$

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$$\sigma_{ij} = 2\mu\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) \tag{3}$$

and kinematic relations between the strain tensor and the displacement vector

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i},_{j} + u_{j},_{i}\right) \tag{4}$$

It is possible to obtain a correct formulation of the problem with respect to the three components of the displacement vector

$$\Delta u_i + \frac{1}{1-2\nu}u_{k},_{ki} + X_i = 0, \tag{5}$$

where m and v are shear modulus and Poisson's ratio, respectively

The equation Lame differentiating with respect to the x_i coordinate and summing up we obtain

$$\Delta u_{k},_{k} = -\frac{1-2\nu}{1-\nu}X_{k},_{k}, \tag{6}$$

This immediately implies the biharmonic nature of the components of the displacement vector:

$$\Delta\Delta u_i + \Delta\left(X_i - \frac{1}{1-\nu}\mathrm{X}_{k},_{ki}\right) = 0 \tag{7}$$

It is not difficult to see that for the symmetric strain tensor (4) the compatibility equations hold:

$$\Delta\varepsilon_{ij} + \varepsilon_{kk},_{ij} - \varepsilon_{ik},_{kj} - \varepsilon_{jk},_{ki} + (\varepsilon_{nk},_{kn} - \Delta\varepsilon_{nn})\delta_{ij} = 0, \tag{8}$$

Equations of state (3) resolving with respect to deformations

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\nu}{1+\nu} \sigma_{kk}\delta_{ij}\right), \tag{9}$$

and substituting in (8) taking into account the equilibrium equation, after some calculations we will have a system of Beltrami-Mitchell equations in stresses:

$$\Delta\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} + X_{i},_{j} + X_{j},_{i} + \frac{\nu}{1-\nu} X_{k},_{k}\delta_{ij} = 0, \tag{10}$$

Convolving the obtained equations, we have:

$$\Delta\sigma_{kk} + \frac{1+\nu}{1-\nu} X_{k},_{k} = 0, \tag{11}$$

Hence, it immediately follows that the symmetric stress tensor is a biharmonic function:

$$\Delta\Delta\sigma_{ij} + \Delta X_{i},_{j} + \Delta X_{j},_{i} + \frac{\nu}{1-\nu}\Delta X_{k},_{k}\delta_{ij} - X_{k},_{kij} = 0\,. \tag{12}$$

Differentiating the system of Beltrami-Mitchell equations -(8) $with\ respect\ to\ the\ x_j\ coordinate$

$$\Delta\left(\sigma_{ij},_{j} + X_i\right) = 0, \tag{13}$$

the biharmonic nature of the equilibrium equations is achieved, on the basis of which, requiring the fulfillment of

$$\left(\sigma_{ij},_{j} + X_i\right)\Big|_G = 0 \tag{14}$$

The equilibrium equation is satisfied over the entire volume of the object under consideration. Moreover, the problem of the theory of elasticity in stresses proposed by-B.E. For the six components of the symmetric stress tensor, we will have six resolving equations – (10) and six boundary conditions (2) and (14).

RESULTS AND DISCUSSIONS

If we assume that there are no mass forces, then (10) can be written in a simplified form:

$$\Delta\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0, \tag{15}$$

Assume that a rectangular Cartesian coordinate system is located at the geometric center of a rectangular parallelepiped.

We denote the coordinate normal to the plane of the plates by – z, we will have - $Ox_1x_2\, z$, then in the indices of the stress tensor we should replace $3 \rightarrow z$, therefore (8) should take the following form

$$\begin{gathered}\Delta\sigma_{ij} + \sigma_{ij},_{zz} + \frac{1}{1+\nu}(\sigma_{kk} + \sigma_{zz}),_{ij} = 0,\\ \Delta\sigma_{zz} + \sigma_{zz},_{zz} + \frac{1}{1+\nu}(\sigma_{kk} + \sigma_{zz}),_{zz} = 0,\\ \Delta\sigma_{iz} + \sigma_{iz},_{zz} + \frac{1}{1+\nu}(\sigma_{kk} + \sigma_{zz}),_{iz} = 0,\end{gathered} \tag{16}$$

On opposite faces along the normal coordinate-z ,the normal tangential mutually balanced loads act - $q^{\pm}(x_1, x_2,), g_i^{\pm}(x_1, x_2,)$. The remaining grains are free from loads. Then the boundary conditions on arbitrary faces of the parallelepiped can be written as follows:

$$\begin{cases} \sigma_{zi} = g_i^{\pm}(x_1, x_2,), \quad \sigma_{zz} = q^{\pm}(x_1, x_2,) \\ \sigma_{ij},_j + \sigma_{iz},_z = 0, \\ \sigma_{zz},_z + g_i^{\pm},_i (x_1, x_2,) = 0, \end{cases} \quad \text{for} \quad z=\pm c \tag{17}$$

$$\begin{cases} \sigma_{11} = 0, \ \sigma_{12} = 0, \quad \sigma_{1z} = 0 \\ \sigma_{ij},_j + \sigma_{iz},_z = 0 \\ \sigma_{zz},_z + \sigma_{zi},_i = 0 \end{cases} \quad \text{for} \quad x_{x1} = \pm l1_1 \tag{18}$$

$$\begin{cases} \sigma_{21} = 0, \ \sigma_{22} = 0, \quad \sigma_{2z} = 0 \\ \sigma_{ij},_j + \sigma_{iz},_z = 0 \\ \sigma_{zz},_z + \sigma_{zi},_i = 0 \end{cases} \quad \text{for} \quad x_{x2} = \pm l2_2 \tag{19}$$

Assumingthat tangential loads are potentiali

$$g_i^{\pm}(x_1, x_2) = g,_i^{\pm}(x_1, x_2) \tag{20}$$

introduce the following voltage potentials: $F_i = F_i(x_1, x_2, z)$, the components of the stress tensor is defined as follows

$$\sigma_{ij} = F_1,_{ij}, \quad \sigma_{zz} = F_2,_{zz}, \quad \sigma_{zi} = F_3,_{zi}, \tag{21}$$

which, substituting in the system of equations (16) and after integration, we have

$$\begin{aligned} &\Delta F_1 + F_1,_{zz} + \frac{1}{1+\nu}(\Delta F_1 + F_2,_{zz}) = 0, \\ &\Delta F_2 + F_2,_{zz} + \frac{1}{1+\nu}(\Delta F_1 + F_2,_{zz}) - \Delta C_u(x_1, x_2) + z\Delta C_v(x_1, x_2) = 0, \\ &\Delta F_3 + F_3,_{zz} + \frac{1}{1+\nu}(\Delta F_1 + F_2,_{zz}) = 0. \end{aligned} \tag{22}$$

Here, $C_u(x_1, x_2), C_v(x_1, x_2)$- are the unknown integration functions to be determined, and the remaining functions and integration constants are considered trivial in the first approximation.

The potentials of the stress tensor and the unknown функции integration functions, decomposing in a series with respect to the basis functions, we have

$$\begin{pmatrix} F_i(x_1, x_2, z) \\ C_u(x_1, x_2) \\ C_v(x_1, x_2) \end{pmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{pmatrix} Z_{inm}(z) \\ Z_{unm} \\ Z_{vnm} \end{pmatrix} \varphi_{nm}(x_1, x_2) \tag{23}$$

and substituting in (13) taking into account the spectral equation

$$\Delta \varphi_{nm} = \gamma_{nm}^2 \varphi_{nm} \tag{24}$$

for the Laplacian, introducing dimensionless coordinates

$$x = \frac{x_1}{l_1}, y = \frac{x_2}{l_2}, \ \xi = \frac{z}{c}. \tag{25}$$

after some calculations, we obtain the final system of resolving equations with respect to the normal coordinate z with respect to the unknowns $Z_{inm}(z)$

$$\begin{aligned} &Z''_{1nm} + \gamma_{nm}^2 \frac{2+\nu}{1+\nu} Z_{1nm} + \frac{1}{1+\nu} Z''_{2nm} = 0, \\ &Z''_{2nm} + \frac{1+\nu}{2+\nu}\gamma_{nm}^2 Z_{2nm} + \frac{1}{2+\nu}\gamma_{nm}^2 Z_{1nm} = \gamma_{nm}^2 (Z_{unm} + c\xi Z_{vnm}), \\ &Z''_{3nm} + \gamma_{nm}^2 Z_{3nm} + \frac{1}{1+}\left(\nu\, \gamma_{nm}^2 Z_{1nm} + Z''_{2nm}\right) = 0, \end{aligned} \tag{26}$$

Gentle external load $g^{\pm}(x_1, x_2,), q^{\pm}(x_1, x_2,)$ laid out on a coordinate functions, Rabreshumi equations (22) and the corresponding boundary conditions (17)-(19) with (24) and (26), after some calculations can be written in the form:

$$\begin{aligned} &Z_{2nm}^{\text{iv}} + 2\gamma_{nm}^2 Z''_{2nm} + \gamma_{nm}^4 Z_{2nm} = \gamma_{nm}^4 (Z_{unm} + c\xi Z_{vnm}), \\ &Z_{1nm} = \frac{2+\nu}{\gamma_{nm}^2}\left[\gamma_{nm}^2 (Z_{unm} + c\xi Z_{vnm}) - Z''_{2nm} - \frac{1+\nu}{2+\nu}\gamma_{nm}^2 Z_{2nm}\right], \\ &Z''_{3nm} + \gamma_{nm}^2 Z_{3nm} - Z''_{2nm} - \gamma_{nm}^2 Z_{2nm} = \frac{1}{1+}\nu\, \gamma_{nm}^2 (Z_{unm} + c\xi Z_{vnm}), \end{aligned} \tag{27}$$

$$\begin{cases} Z''_{2nm} = q_{nm}^{\pm}, \\ Z''_{3nm} + \gamma_{nm}^2 Z_{1nm} = 0, \\ Z'_{3nm} = g_{nm}^{\pm}, \\ Z_{3nm} = -\frac{q_{nm}^{\pm}}{\gamma_{nm}^2}, \end{cases} \quad \text{when} \quad \xi=\pm 1 \tag{28}$$

$$\varphi_{nm},_{111} = 0, \ \ \varphi_{nm},_{11} = 0, \ \varphi_{nm},_{12} = 0, \ \ \varphi_{nm},_1 = 0, \quad \text{when} \quad x = \pm 1 \tag{29}$$

$$\varphi_{nm},_{222}=0,\ \varphi_{nm},_{22}=0,\ \varphi_{nm},_{12}=0,\ \varphi_{nm},_{2}=0, \qquad \text{for} \qquad y=\pm 1 \tag{30}$$

We represent the basis functions as cosine polynomials

$$\varphi_{nm}(\mathrm{x},\mathrm{y})=\left[cos\pi nx-\tfrac{1}{9}cos3\pi nx\right]\left[cos\pi my-\tfrac{1}{9}cos3\pi my\right] \tag{31}$$

As follows from the representation of the basis functions (31), complete satisfaction of (29)-(30) is achieved.

In the spectral problem (16), we apply the method Bubnova-Galerkin. Atthe same time, the corresponding basis function sand with their derivatives in the form of cosine polynomials are mutually orthogonal, hence infinite trigonometric are convergent. In this case, we use the Rayleigh-Ritz formula to determine the eigenvalues:

$$\gamma_{nm}^2=\frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\iint_G \Delta\varphi_{kl}\varphi_{nm}dG}{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\iint_G \varphi_{kl}\varphi_{nm}dG} \tag{32}$$

In the problem under consideration, taking into account (31) for the eigenvalues, we have the following expressions:

$$\gamma_{nm}^2=-5\frac{5}{9}\pi^2c^2\left(\frac{n^2}{l_1^2}+\frac{m2^2}{l_2^2}\right)=-\mu_{nm}^2 \tag{33}$$

After determining the multiple roots of the characteristic equation corresponding to the first equation (27) for Z_{inm}, we haveиthe followingяsolutions:

$$\begin{cases} Z_{1nm}=Z_{2nm}+2\frac{2+\nu}{\mu_{nm}}c(V_{1nm}sh\mu_{nm}\xi+V_{2nm}ch\mu_{nm}\xi),\\ Z_{2nm}=(U_{1nm}+\mathrm{c}\xi V_{1nm})ch\mu_{nm}\xi+(U_{2nm}+\mathrm{c}\xi V_{2nm})sh\mu_{nm}\xi+(Z_{unm}+c\xi Z_{vnm})\\ Z_{3nm}=U_{3nm}ch\mu_{nm}\xi+U_{4nm}sh\mu_{nm}\xi+c\xi(V_{1nm}sh\mu_{nm}\xi+V_{2nm}ch\mu_{nm}\xi)+\frac{1}{1+\nu}(Z_{unm}+c\xi Z_{vnm}) \end{cases} \tag{34}$$

Consider a special case, $q_{nm}^{\pm}=q_{nm}, g_{nm}^{\pm}=0$. Substituting (34) into (28), determineм the unknown integration constants

$$U_{1nm}=\frac{q_{nm}}{\mu_{nm}^2ch\mu_{nm}}-\left(th\mu_{nm}+\frac{2}{\mu_{nm}}\right)cV_{2nm},\quad cV_{1nm}=-\frac{th\mu_{nm}}{1+\frac{2th\mu_{nm}}{\mu_{nm}}}U_{2nm},$$

$$Z_{unm}=(1+\nu)\left[\frac{q_{nm}}{\mu_{nm}^2}+\frac{\mu_{nm}+th\mu_{nm}-th\mu_{nm}sh\mu_{nm}}{\mu_{nm}+2th\mu_{nm}}U_{2nm}\right],$$

$$cZ_{vnm}=-(1+\nu)\frac{1+\mu_{nm}+\mu_{nm}th^2\mu_{nm}-2\mu_{nm}th\mu_{nm}}{\mu_{nm}-\mu_{nm}th\mu_{nm}}ch\mu_{nm}cV_{2nm},$$

$$U_{4nm}=\frac{1-\mu_{nm}+\mu_{nm}th\mu_{nm}}{\mu_{nm}-\mu_{nm}th\mu_{nm}}cV_{2nm},\quad U_{3nm}=\frac{\mu_{nm}+th\mu_{nm}}{\mu_{nm}+2th\mu_{nm}}U_{2nm},$$

$$cV_{2nm}=\frac{B}{A}U_{2nm},\qquad U_{2nm}=-\frac{1}{2ch\mu_{nm}}\frac{q_{nm}}{C\mu_{nm}^2}$$

Where

$$\mathrm{A}=\mu_{nm}-2\mu_{nm}th\mu_{nm}+3th\mu_{nm}+(\mu_{nm}-2)th^2\mu_{nm}-(1+\nu)(1+\mu_{nm}+\mu_{nm}th^2\mu_{nm}-2\mu_{nm}th\mu_{nm}),$$

$$\mathrm{B}=\left[\frac{th\mu_{nm}}{1+\frac{2th\mu_{nm}}{\mu_{nm}}}+\frac{2(2+\nu)}{\mu_{nm}}\frac{th^2\mu_{nm}}{1+\frac{2th\mu_{nm}}{\mu_{nm}}}-th\mu_{nm}\right](1-th\mu_{nm}),$$

$$\mathrm{C}=\frac{B}{A}-\frac{1}{2(1+\nu)}\left[1-\frac{th^2\mu_{nm}}{\mu_{nm}^2}-\frac{th\mu_{nm}}{\mu_{nm}}+\frac{1+\nu}{ch\mu_{nm}}\left(1+\frac{th\mu_{nm}}{\mu_{nm}}-\frac{th\mu_{nm}}{\mu_{nm}}sh\mu_{nm}\right)\right]\frac{1}{1+2\frac{th\mu_{nm}}{\mu_{nm}}},$$

Now let us consider the limiting cases of a parallelepiped, namely, an infinite reduction in size in the direction of action of a mutually balanced normal load, i.e. $\mathrm{c}\to 0$,from which $\mu_{nm}\to 0$, immediately follow,
From here we will have the required solutions in the following form.

$$\sigma_{ij}\to\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}2\left[\frac{1-(2+\nu)\frac{th\mu_{nm}}{\mu_{nm}}\left(1-\frac{\nu}{2+\nu-\mu_{nm}^2}\right)}{1-th^2\mu_{nm}+\frac{th\mu_{nm}}{\mu_{nm}}}\frac{q_{nm}}{\mu_{nm}^2}-\xi\frac{(2+\nu)\left(1-\frac{\nu}{(2+\nu)\left(1+\frac{th\mu_{nm}}{\mu_{nm}}\right)-\mu_{nm}^2}\right)}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\frac{th^2\mu_{nm}}{\mu_{nm}^2}\mathrm{c}\frac{g_{nm}}{\mu_{nm}}\right]\varphi_{nm}(x,y),_{ij}\approx\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{-\frac{q_{nm}}{\mu_{nm}^2}+\xi\frac{4+\nu}{\mu_{nm}-th\mu_{nm}}\mathrm{c}g_{nm}\right\}\varphi_{nm}(x,y),_{ij}\ .$$

$$\sigma_{zz} \to \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{\frac{ch\mu_{nm}}{ch\mu_{nm}}+\frac{1}{\mu_{nm}}th\mu_{nm}\frac{ch\mu_{nm}}{ch\mu_{nm}}-\xi^2 th\mu_{nm}\frac{sh\mu_{nm}}{ch\mu_{nm}}}{1-th^2\mu_{nm}+\frac{th\mu_{nm}}{\mu_{nm}}}q_{nm}+\xi\frac{\frac{sh\mu_{nm}}{ch\mu_{nm}}-\frac{ch\mu_{nm}}{ch\mu_{nm}}th\mu_{nm}}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\mu_{nm}\mathrm{c}g_{nm}\right\}\varphi_{nm}(x,y) \approx$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1+\frac{th\mu_{nm}}{\mu_{nm}}-\xi^2 th^2\mu_{nm}}{1-th^2\mu_{nm}+\frac{th\mu_{nm}}{\mu_{nm}}}q_{nm}+\xi\frac{\frac{sh\mu_{nm}}{ch\mu_{nm}}-th\mu_{nm}}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\mu_{nm}\mathrm{c}g_{nm}\right\}\varphi_{nm}(x,y) \approx \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_{nm}\,\varphi_{nm}(x,y) = q(x,y).$$

$$\sigma_{iz} \to \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\xi\frac{\frac{sh\mu_{nm}}{ch\mu_{nm}}-\frac{ch\mu_{nm}}{ch\mu_{nm}}th\mu_{nm}}{1-th^2\mu_{nm}+\frac{th\mu_{nm}}{\mu_{nm}}}\frac{q_{nm}}{\mu_{nm}}+\frac{\frac{ch\mu_{nm}}{ch\mu_{nm}}-\xi^2\frac{sh\mu_{nm}}{ch\mu_{nm}}th\mu_{nm}-\frac{ch\mu_{nm}}{\mu_{nm}ch\mu_{nm}}th\mu_{nm}}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\mathrm{c}g_{nm}-\right.$$

$$\left.\frac{2\nu}{\left(1+\frac{th\mu_{nm}}{\mu_{nm}}\right)-\frac{\mu_{nm}^2}{2+\nu}}\frac{\frac{ch\mu_{nm}}{ch\mu_{nm}}-1}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\frac{th^2\mu_{nm}}{\mu_{nm}^2}\mathrm{c}g_{nm}\right\}\varphi_{nm}(x,y)_{,i} \approx$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1-\xi^2 th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}{1-th^2\mu_{nm}-\frac{th\mu_{nm}}{\mu_{nm}}}\mathrm{c}g_{nm}\,\varphi_{nm}(x,y)_{,i} \approx \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathrm{c}g_{nm}\,\varphi_{nm}(x,y)_{,i} = cg(x,y)_{,i}.$$

CONCLUSIONS

- The correctness of fitting spatial problems of the theory of elasticity in stresses is shown
- For special mutually balanced static loads, the problem is reduced to solving three equations in new stress potentials
- Thus, a numerically-analytically approach to solving the harmonic equation of the spatial problem of the theory of elasticity in stresses has been developed. The results are compared with the data of finite element analysis using the ANSYS program and the data obtained by A.V. Alexandrov [18]

REFERENCES

1. A. N. Konavalov Numerical methods in dynamic problems of elasticity theory, Sib. Matematicheskiy zhurnal, 1997, vol. 38, no. 3, pp. 551-568
2. Khudoynazarov Kh.Kh., Khalmuradov R.I., Yalgashev B.F. 2021 Longitudinal-radial vibrations of a elastic cylindrical shell filled with a viscous compressible liquid. *Tomsk State University. Journal of Mathematics and Mechanics*. 69, 139-154. https://doi.org/10.17223/19988621/69/11
3. Demidov, S. P. (1979). Teoriya uprugosti. M.: Vyisshaya shkola, 432.
4. Kayumov, R.A. (2016). Osnovyi teorii uprugosti i elementyi teorii plastin i obolochek: Uchebnoe posobie. Kazan: Izd-vo Kazansk. gos. arhitek.-stroit. un-ta, 111.
5. Zamyatin, V.M., Mahov, A.V., Svetashkov, A.A. (2006). Reshenie ploskih zadach teorii uprugosti dlya polosyi s pomoschyu diagonalizovannoy sistemyi uravneniy ravnovesiya. Izvestiya Tomskogo politehnicheskogo universiteta, 309 (6), 135-139.
6. Tokovyi, Yu. V., Rychahivs'kyi, A. V. (2005). Analytic Solution of the Plane Problem of the Theory of Elasticity for a Nonuniform Strip. Materials Science, 41 (1), 135-138.
7. Li, Lian He; Fan, Tian You (2008). Complex variable function method for the plane elasticity and the dislocation problem of quasicrystals with point group 10 mm. Physics Letters A, 372 (4), 510-514.
8. Elakkad, A., Bennani, M.A., Mekkaoui, J. EL, Elkhalfi, A. (2013). A Mixed Finite Element Method for Elasticity Problem. International Journal of Advanced Computer Science and Applications, 4 (2), 161-166.
9. B. E. Pobedrya On the static problem in adverbs, Vestnik MSU. Ser. Matem., mekh., 2003, No. 3, pp. 61-67
10. Daschenko, A.F., Kolomiets, L.V., Orobey, V.F., Surianinov, M.G. (2010) Chislenno– analiticheskiy metod granichnyih elementov. Odessa: VMV, 1, 416, 2, 512.
11. Orobey, V.F., Surianinov, M.G. (2011). Osnovnyie polozheniya chislennoanaliticheskogo varianta MGE. St. Petersburg: St. Petersburg Polytechnic Universities Press, 4 (22), 33-39.
12. A.V.Aleksandrov, V.D. Potapov (1990). Osnovyi teorii uprugosti i plastichnosti. M.: Vyisshaya shkola, 398.
13. T. Kholmatov On methods for solving the problem in stresses, DAN SSSR, 1980, vol. 252 No. 2, pp. 315-317

14. Simona De Cicco, Fabio De Angelis. A plane strain problem in the theory of elastic materials with voids. Mathematics and Mechanics of Solids. 2020; 25(1): 46-59.
15. Yu.Ya. Tyukalov Equilibrium finite elements for plane problems of the elasticity theory. Magazine of Civil Engineering. 2019. 91(7). Pp. 80–97. DOI: 10.18720/MCE.91.8
16. Akhmedov A.B., Sheshenin, S.V. Nonlinear equations of motion for orthotropic plates. MoscowUniv. Mech.Bull.67,66-68 (2021). https://doi.org/10.3103/S002713301203003X
17. Rustam Khalmuradov and Utkir Nishonov Nonlinear deformation of circular discrete ribbed plate under of pulse loading. 2021 *E3S Web of Conferences* **264** 02018. https://doi.org/10.1051/e3sconf/202126402018
18. M. Svanadze Steady vibration problems in the theory of elasticity for materials with double voids. Acta Mech 2018; 229(4): 1517–1536.