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Numerical Implementation of Spectral Problems for Non-Self-Adjoint Operators

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Abstract. In engineering applications for determining natural frequencies during vibrations and critical forces in stability problems, an important place is occupied by the numerical implementation of spectral problems. In this paper, we propose a numerical approach to solving the differential problem in spectral problems for non-self-adjoint operators. As an example, the problem of determining critical compressive forces and torques in rods is considered.

INTRODUCTION

A number of engineering problems are reduced to considering a system of spectral equations that have a unique solution only if the value of some parameter included in them is known. This parameter is called the characteristic or eigenvalue of the system. At the same time, when solving spectral problems, depending on the consideration of specific applied problems, mathematical difficulties of an implementation nature arise. In this regard, spectral problems are classified as fundamental problems of mathematical physics.

The solution of spectral problems for given coordinate functions is reduced to the algebraic problem of determining the values of eigenvalues using characteristic equations. In this case, one often has to deal with ill-conditioned matrices of resolving algebraic equations, which is undesirable for determining the values of higher frequencies of natural oscillations.

The fundamental work of Wilkinson [1] is devoted to the algebraic problem, where the main aspects of this issue are investigated.

In order to overcome the arising difficulties, instead of solving the algebraic problem of determining the eigenvalues, we consider the solution of a similar differential problem in spectral problems. In this case, the basis functions are not given a priori, but are defined as a solution to homogeneous boundary value problems. In the fundamental work of Collatz [2], the main approaches to solving the differential problem for self-adjoint operators of resolving equations in dynamic problems of applied mechanics are considered. It is well known that if a continuous operator is self-adjoint, then the system of its root vectors is complete in the range of this operator. What cannot be asserted for non-self-adjoint operators [3]. Therefore, for non-self-adjoint operators, questions of the completeness of functional infinite series with respect to basic functions become of particular interest.

The theory of non-self-adjoint operators is necessary for the mathematical study of processes that arise in non-conservative systems, which play an important role in modern physics and mechanics. It has recently attracted more and more attention of mathematicians and physicists, and sometimes engineers [4].

In contrast to the theory of self-adjoint operators, in the abstract theory of non-self-adjoint operators, before 1950, no spectral expansions and even theorems on the completeness of the system of root vectors were obtained.

This situation changed only in 1951 due to advances in the abstract theory of non-self-adjoint operators. In that year, the work of M.V. Keldysh [5], in which he established theorems on the completeness of root vectors and theorems on the asymptotic properties of eigenvalues for a wide class of polynomial pencils of non-self-adjoint operators. These theorems made it possible to obtain important results in boundary value problems for partial differential equations.

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The papers [6-9] propose numerical approaches for solving the differential problem in spectral problems.

SOLUTION METHOD

A one-dimensional unified spectral problem can be written as systems of differential equations in vector-matrix form [6]:

$$[\mathrm{C(x)}\mathbf{V}'+\mathrm{Q(x)}\mathbf{V}]'+\mathrm{A(x)}\mathbf{V}'-\mathrm{pB(x)}\mathbf{V}=0 \tag{1}$$

$$a_x\mathbf{V}'+\mathrm{b_x}\,\mathbf{V}=0 \quad at \quad x=0,l \tag{2}$$

where, C(x), Q(x), A(x), B(x) – matrix-functions, a_x, b_x – constant matrices, the form of which depends on the problems under consideration. Here the eigenvalue p is determined from the condition of non-triviality of the solution of the homogeneous boundary value problem (1) - (2).

This boundary value problem is solved by the matrix differential sweep method [9], for which the following system of homogeneous ordinary differential equations is introduced into consideration:

$$\alpha[\mathrm{C(x)}\mathbf{V}'+\mathrm{Q(x)}\mathbf{V}]+\mathrm{b}\,\mathbf{V}=0 \tag{3}$$

where, matrix functions - α, β are defined as the solution of the following inhomogeneous Cauchy problem for matrix functions:

$$\begin{cases}\alpha'=[\alpha\,\mathrm{A(x)}-\beta]\mathrm{C}^{-1}(\mathrm{x}) & \alpha(0)=\mathrm{a}_0\mathrm{C}^{-1}(0)\\ \beta'=\alpha\,\mathrm{B(x)}-\mathrm{Q(x)} & \beta(0)=\mathrm{b}_0-\mathrm{a}_0\mathrm{C}^{-1}(0)\,\mathrm{Q}(0)\end{cases} \tag{4}$$

After determining the required unknowns at the points $x = l$, taking into account (2) and (3), we will have the following system of 2 × n linear algebraic equations with respect to the unknowns $\mathbf{V}(l)$,$\mathbf{V}'(l)$:

$$\begin{pmatrix}\alpha(\mathrm{l})\,\mathrm{C(l)} & \beta(\mathrm{l})+\alpha(\mathrm{l})\,\mathrm{Q(l)}\\ \mathrm{a}_l & \mathrm{b}_l\end{pmatrix}\begin{pmatrix}\mathbf{V}'\\ \mathbf{V}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5}$$

which implicitly depends on the eigenvalue *p*. It, in turn, is determined from the nontriviality condition for solution (5):

$$S(p)=\det\begin{pmatrix}\alpha(\mathrm{l})\,\mathrm{C(l)} & \beta(\mathrm{l})+\alpha(\mathrm{l})\,\mathrm{Q(l)}\\ \mathrm{a}_l & \mathrm{b}_l\end{pmatrix}=0 \tag{6}$$

By substituting the obtained values of the equation roots (6) - p_x into (5), we can calculate the values of the basic functions $\mathbf{V}'_{\mathrm{k}}(\mathrm{l}), \mathbf{V}_{\mathrm{k}}(\mathrm{l})$. Further, using these values, the basic functions on the segment $x\in[l,0]$ are additionally determined by the reverse course of the matrix method of differential sweep. The above scheme is suitable for the analytical solution of the spectral problem. And in the numerical implementation, an iteration process of Wegstein [10,11] is constructed to determine the eigenvalues:

$$\begin{gathered}\omega_1=p_1=S(p_0),\ \ p_{n+1}=S(\omega_n),\\ \omega_{n+1}=p_{n+1}-\frac{(p_{n+1}-p_n)(p_{n+1}-\omega_n)}{p_{n+1}-p_n+\omega_{n+1}-\omega_n)}\end{gathered} \tag{7}$$

The condition for terminating the iterative process is:

$$S(\omega_n)<\varepsilon \tag{8}$$

As it is known, the positive definiteness and self-ad jointness of the operator of resolving equations of boundary value problems of applied mechanics is achieved for differential equations of mathematical physics of even order with basic or natural boundary conditions. In this case, the positive definiteness of the eigenvalues and the orthonormality of the basic functions in spectral problems is rigorously proved. At the same time, when solving applied problems of mechanics, cases of non-self-ad jointness of differential operators of resolving equations arise.

The complexity of solving spectral problems, among other things, is due to the fact that in some cases the eigenvalue can also participate in the boundary conditions. For example, with simultaneous compression and torsion of a rod, we will have the following spectral problem [2]:

$$[D(DU'')']' + P[(DU')' + DU''] + M^2U' + P^2U = 0 \tag{9}$$

with boundary conditions:

$$U = (DU'')' + PU' + M^2U'/D = 0 \text{ at } x = 0, l \tag{10}$$

where D – is the bending stiffness of the rod, P is the longitudinal compressive force, M is the external torque, U – is the displacement of the structure. In this case, the critical value of both the compressive force - P and the torque M can be used as eigenvalues. In this problem, for the compressive force P, we have a parabolic dependence. Here the resolving equation contains the convective term - M^2U', while the eigenvalues also participate in the boundary conditions. Therefore, for this problem, the operator of the resolving equation is non-self-adjoint.

By introducing the notation $W = DU''$, instead of the spectral problem for a fourth-order ordinary differential equation, we obtain an equivalent system of second-order ordinary differential equations:

$$\begin{cases} (DW')' + P(2W + D'U') + M^2U' + P^2U = 0 \\ U'' - \dfrac{W}{D} = 0 \end{cases} \tag{11}$$

$$\begin{cases} W' + \left(P + \dfrac{M^2}{D}\right)U' = 0 \\ U = 0 \end{cases} \tag{12}$$

If in the spectral problem (1) and (2), as matrices of functions and matrices of constants, we set:

$$C = \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix},\ Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & PD' + M^2 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2P & M^2 \\ 0 & -\dfrac{1}{D} \end{pmatrix},$$

$$a_0 = a_1 = \begin{pmatrix} 1 & P + \dfrac{M^2}{D} \\ 0 & 0 \end{pmatrix},\ b_0 = b_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ V = \begin{pmatrix} W \\ U \end{pmatrix}$$

then it becomes equivalent to problems (11) and (12).

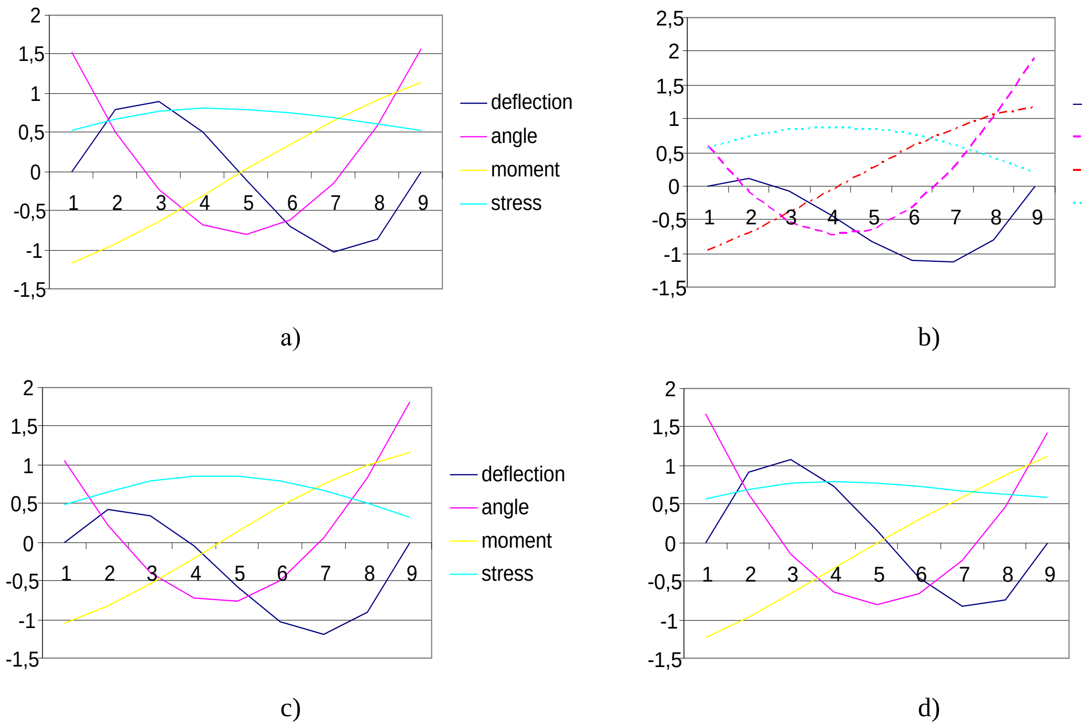
To solve the unified spectral problem, a package of application programs for solving boundary and spectral problems based on the matrix method of differential sweep in the Turbo-Pascal language in the integrated Windows-10 environment has been compiled.

THE DISCUSSION OF THE RESULTS

As it is known, the spectral problems (11) and (12) take place in drilling wells in the oil and gas industry. In wells, drilling equipment is subjected to a combination of compressive loads and torques. When their critical values are reached, an emergency situation occurs in drilling rigs. In this regard, the determination of the critical values of the compressive force - P_k torque- M_k is of practical importance.

As follows from the results obtained, with an increase in the impact of torque, the critical value of the compressive force decreases, but on the other hand, by reducing the compressive force, it is possible to increase the critical value of the torque. These facts must be taken into account in the process of developing oil and gas fields with complex mining and geological structures. For example, in loess deposits, the drilling rig can be operated with a high compressive load at low RPM.

A feature of the numerical solution of the differential problem of spectral problems turned out to be the fact that even under symmetric boundary conditions, which takes place in the problem under consideration, the form of the basic functions is asymmetric along the coordinate (Fig. 1).



a) P= 1,474, M=2,5. b) P= 2,443, M=0. c) P= 2,306, M=1. d) P= 1,045, M=2,9.

FIGURE 1. Influence of the joint action of the compressive force - P and torque - M on the nature of the distribution of basic functions

As can be seen from the graphs, for various combinations of critical values of torque and compressive force, a qualitative change occurs in the equilibrium shapes for the rods. Thus, when solving spectral problems for non-self-adjoint operators of resolving equations, the choice of symmetric basic functions is not always justified. These facts take place in solving the algebraic problem of determining the eigenvalues for given basic functions. In this regard, for non-self-adjoint operators, the numerical solution of the differential problem in spectral problems becomes a promising direction.

MAIN CONCLUSIONS

1. The numerical method of differential sweep turned out to be a fairly effective approach for solving the differential problem in spectral problems, including for non-self-adjoint operators.
2. The proposed approach also successfully solves spectral problems in the presence of eigenvalues and in boundary conditions.
3. The problems of stability of rods under the combined action of compressive forces and torques are considered.

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